Inverse Problems in the Estimation of the Lévy-Triplet of Infinitely Divisible Random Fields

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δ-Ring

Definition (δ-Ring)

A (non-empty) system of sets $\mathcal{R}$ is called a δ-ring, if for arbitary sets $A, B \in \mathcal{R}$ and any sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{R}$ it holds:

1. $A \cup B \in \mathcal{R}$
2. $A \backslash B = A \cap B^c \in \mathcal{R}$
3. $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{R}$

Example

- $\mathcal{R} = \{A \in \mathcal{B}(\mathbb{R}^d) : \; A \text{ bounded}\} =: \mathcal{B}_0(\mathbb{R}^d)$
- $\mathcal{R} = \{A \in \mathcal{B}(\mathbb{R}^d) : \; \nu_d(A) < \infty\}$, ν_d-Lebesgue-measure in $\mathbb{R}^d$

ID Random Measures

Let $D \neq \emptyset$ be a non-empty set and $\mathcal{D}$ a δ-ring of subsets of D, such that $\exists$ an increasing sequence $\{D_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$ with

$$\bigcup_{n=1}^{\infty} D_n = D$$

Definition (ID random measure)

A stochastic process $\Lambda = \{\Lambda(A),\ A \in \mathcal{D}\}$ is called **ID** random measure , if for any sequence of pairwise disjoint sets $\{E_n\}_{n\in\mathbb{N}}$ it holds:

1. $\Lambda(E_n)$, $n = 1, 2, \ldots$ are independent random variables (independently scattered)
2. $\Lambda(\bigcup_{n=1}^{\infty} E_n) \overset{f.s.}{=} \sum_{n=1}^{\infty} \Lambda(E_n)$, if $\bigcup_{n=1}^{\infty} E_n \in \mathcal{D}$ (σ-additivity)
3. $\Lambda(A)$ is an **ID** random variable for every set $A \in \mathcal{D}$

For $A \in \mathcal{D}$ the characteristc function $\varphi_{\Lambda(A)}$ of $\Lambda(A)$ is given by

$$\varphi_{\Lambda(A)}(z) = \exp\left\{ iz\zeta_0(A) - \frac{1}{2}z^2\zeta_1(A) + \int_{\mathbb{R}} (e^{izx} - 1 - izx\tau(x))v_A(dx) \right\}$$

where $-\infty < \zeta_0(A) < \infty$, $0 \leq \zeta_1(A) < \infty$ and v_A denotes the Lévy-measure. The function $\tau : \mathbb{R} \to \mathbb{R}$ is defined by

$$\tau(x) = \begin{cases} 1 & ; |x| \leq 1 \\ \frac{1}{|x|} & ; |x| > 1 \end{cases}$$

The continuation of measure $\lambda : \mathcal{D} \to [0, \infty)$ to $\sigma(\mathcal{D})$, defined by

$$\lambda(A) = |\zeta_0|(A) + \zeta_1(A) + \int_{\mathbb{R}} \min\{1, x^2\}\nu_A(dx), \;\; A \in \mathcal{D},$$

is referred to as control measure.

Examples

- Stable Random Measures:
  $\beta : \mathbb{R}^d \to [-1, 1]$ measurable (skewness intensity), $M = \{M(A),\ A \in \mathcal{B}_0(\mathbb{R}^d)\}$ **ID** random measure with

$$M(A) \sim S_\alpha((\nu_d(A))^{1/\alpha}, \frac{\int_A \beta(x)dx}{\nu_d(A)}, 0).$$

- Poisson Random Measures:
  $\Theta : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$, $\Theta(A) < \infty$, for every $A \in \mathcal{B}_0(\mathbb{R}^d)$,
  $\psi = \{\psi(A),\ A \in \mathcal{B}_0(\mathbb{R}^d)\}$ **ID** random measure with

$$\psi(A) \sim Poi(\Theta(A)).$$

Integration w.r.t. **ID** Measures

Definition (**ID** stochastic integral)

Let Λ be an **ID** random measure and $A \in \sigma(\mathcal{D})$.

1. For a simple function $f : D \to \mathbb{R}$, $x \mapsto \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}(x)$ the **ID** integral is defined by
$$\int_A f(x)\Lambda(dx) = \sum_{i=1}^{n} x_i \Lambda(A \cap A_i)$$
$n \in \mathbb{N}$, $A_1, \dots, A_n \in \mathcal{D}$ pairwise disjoint, $x_1, \dots, x_n \in \mathbb{R}$.
2. For a Λ-integrable function $f : D \to \mathbb{R}$ one defines
$$\int_A f(x)\Lambda(dx) = \operatorname*{plim}_{n\to\infty} \int_A f_n(x)\Lambda(dx)$$
with a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions, such that $f_n \overset{n\to\infty}{\longrightarrow} f$, λ a.e.

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Setting

Let $X = \{X(t);\ t \in T\}$, $T \subset \mathbb{R}^d$ be a stationary random field with an integral representation

$$X(t) = \int_{\mathbb{R}^d} f(x-t)\Lambda(dx), \quad t \in T, \tag{1}$$

where $\Lambda = \{\Lambda(A);\ A \in \mathcal{B}_0(\mathbb{R}^d)\}$ is a homogeneous infinitely divisible (**ID**) random measure and f is a deterministic Λ-integrable function. The above integral is understood as an **ID** stochastic integral. Let

$$\varphi_{\Lambda(A)}(z) = \exp\left\{ iz\zeta_0(A) - \frac{z^2}{2}\zeta_1(A) + \int_{\mathbb{R}} \left(e^{ixz} - 1 - izx\tau(x)\right) v_A(dx) \right\},$$

$z \in \mathbb{R},\ A \in \mathcal{B}_0(\mathbb{R}^d)$ be the characteristic function of $\Lambda(A)$ with $\tau : \mathbb{R} \to \mathbb{R}$ being defined as

$$\tau(x) = \begin{cases} 1 & , \text{ if } |x| \leq 1 \\ 1/|x| & , \text{ if } |x| > 1. \end{cases}.$$

Setting (continued)

The random measure Λ is assumed to be homogeneous, i.e. we have

$$\zeta_0(du) = a_0 du, \quad \zeta_1(du) = b_0 du, \quad v_A(du) = V_0(du) \cdot \nu_d(A),$$

for each bounded Borel set A, where $a_0 \in \mathbb{R}$, $b_0 \geq 0$ and V_0 is a Lévy measure on $\mathbb{R}$. ν_d denotes the d-dimensional Lebesgue measure. Furhtermore it is assumed, that

$$V_0(du) = v_0(u)du,$$

i.e. V_0 is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^d$. For abbreviation in the following we denote by (a_0, b_0, v_0) the characteristic triplet of $\Lambda(A)$.

Notice that X is an **ID** random field since Λ is **ID**.

Inverse Estimation Problem

Consider observations $X(t_1), \dots, X(t_l)$ of the field in (1), $l \in \mathbb{N}, \ t_1, \dots, t_l \in T$.

How to estimate the characteristic triplet (a_0, b_0, v_0) of Λ given the Lévy characteristics of $X(0)$?

$$X(t) = \int f(x-t)\Lambda(dx) \quad \overset{?}{\longmapsto} \quad \Lambda$$

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Plug-In Approach

Now assume

1. bounded Borel sets $\Delta_1, \dots, \Delta_m$ with $\nu_d(\Delta_k) = \nu_d(\Delta_1)$, for all k
2. $f(x) = \sum_{k=1}^{m} f_k \mathbf{1}_{\Delta_k}(x)$, $x \in \mathbb{R}^d$ a simple function with coefficients $f_k \in [-1, 1] \backslash \{0\}$ for all k.

Under these assumptions one can easily show, that the characteristic triplet (a_1, b_1, V_1) of $X(0)$ has the following representation:

$$a_1 = a_0 \sum_{k=1}^{m} f_k + \nu_d(\Delta_1) \sum_{k=1}^{m} \frac{1}{f_k} \int_{\mathbb{R}} x \mathbf{1}_{[-1,1] \backslash [-|f_k|, |f_k|]}(x) v_0\left(\frac{x}{f_k}\right) dx \tag{2}$$

$$b_1 = b_0 \sum_{k=1}^{m} f_k^2 \tag{3}$$

$$v_1(x) = \nu_d(\Delta_1) \sum_{k=1}^{m} \frac{1}{f_k} v_0\left(\frac{x}{f_k}\right) \tag{4}$$

where $V_1(dx) = v_1(x)dx$.

Plug-In Approach

Iterating these relations one gets

$$v_0(x) = \frac{1}{\nu_d(\Delta_1)}\left(f_1 v_1(f_1 x) + \sum_{k=1}^{\infty}(-1)^k \sum_{i_1,\dots,i_k=2}^{m} \frac{f_1^{k+1}}{f_{i_1}\cdots f_{i_k}} v_1\left(\frac{f_1^{k+1}}{f_{i_1}\cdots f_{i_k}} x\right)\right) \qquad (5)$$

by solving (4) recursively, provided that this series converges.

Lemma

The series in (5) converges absolutely pointwise for all $x \in \mathbb{R}\setminus\{0\}$, if

$$|f_1| > \max\{|f_2|,\dots,|f_m|\} \quad \text{and} \quad v_1(x) = \mathcal{O}(|x|^{-\alpha}), \qquad (6)$$

where $\alpha > 1 + \frac{\log(m-1)}{\log(|f_1|/\max\{|f_2|,\dots,|f_m|\})}$. The convergence is furthermore uniform on every compact interval, which does not contain zero.

Now let $\hat{f}_1, \ldots, \hat{f}_m$ and $\hat{v}_1$ be estimators for the coefficients $f_1, \ldots, f_m$ and the Lévy density v_1. Then the relation (5) leads to the following plug-in estimator $\hat{v}_0$ for v_0:

$$\hat{v}_0(x) = \frac{1}{\nu_d(\Delta)} \left[\hat{f}_1 \hat{v}_1(\hat{f}_1 x) + \sum_{k=1}^{n_l} (-1)^k \sum_{i_1,\ldots,i_k=2}^{m} \frac{\hat{f}_1^{k+1}}{\hat{f}_{i_1} \cdots \hat{f}_{i_k}} \hat{v}_1 \left(\frac{\hat{f}_1^{k+1}}{\hat{f}_{i_1} \cdots \hat{f}_{i_k}} x \right) \right] \quad (7)$$

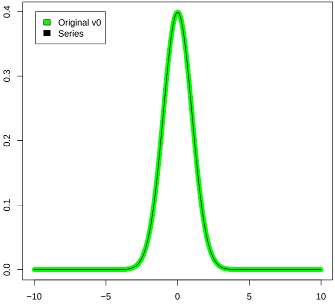
with $\{n_l\}$ being a sequence that grows to infinity with the sample size l going to infinity. By relations (2) and (3) one can easily obtain estimators $\hat{a}_0$ and $\hat{b}_0$ for a_0 and b_0 substituting $a_1,\ b_1,\ v_0$ by their estimators $\hat{a}_1,\ \hat{b}_1,\ \hat{v}_0$. It turned out that the estimator (7) is sensitive to noise and outliers.

Example

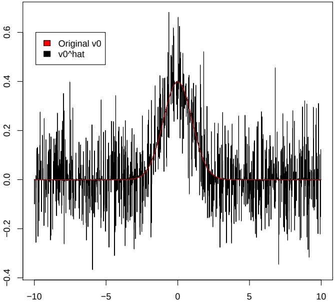
1. $m = 9,\ f_1 = 0.5,\ f_2 = \cdots = f_9 = 0.0625,\ \hat{f}_k = f_k$, for all $k = 1, \ldots, 9$
2. $v_1(x) = \frac{8}{0.0625\sqrt{2\pi}} e^{-\frac{x^2}{2\cdot 0.0625^2}} + \frac{1}{0.5\sqrt{2\pi}} e^{-\frac{x^2}{2\cdot 0.5^2}},\ x \in \mathbb{R}$
3. $v_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},\ x \in \mathbb{R}$

Now set $\hat{v}_1(x) = v_1(x) + \varepsilon(x),\ x \in \mathbb{R}$, with $\varepsilon(x) \sim N(0, \sigma^2)$, for all $x \in \mathbb{R}$, being a Gaussian Noise and choose $n_l = 3$.

Plug-In Approach



(a) v_0 and the series (5).



(b) v_0 and $\hat{v}_0$, $\sigma = 0.000001$.

Figure : Example

It turned out that the estimator (7) is sensitive to noise and outliers. For this reason we consider the estimator $\tilde{v}_0$ defined by

$$\tilde{v}_0(x) = \left[\hat{v}_0 * \psi_h\right](x)$$

and

$$\tilde{\tilde{v}}_0(x) = \frac{1}{\nu_d(\Delta)} \left[\hat{f}_1 \left[\hat{v}_1 * \psi_h\right](\hat{f}_1 x) + \sum_{k=1}^{n_l} (-1)^k \sum_{i_1,\ldots,i_k=2}^{m} \frac{\hat{f}_1^{k+1}}{\hat{f}_{i_1} \cdots \hat{f}_{i_k}} \left[\hat{v}_1 * \psi_h\right] \left(\frac{\hat{f}_1^{k+1}}{\hat{f}_{i_1} \cdots \hat{f}_{i_k}} x \right) \right]$$

where ψ_h denotes a kernel function with bandwidth $h > 0$. Instead of pointwise convergence of $\tilde{v}_0$ we want to prove consistency results in $L_1(\mathbb{R}\backslash[-\varepsilon, \varepsilon])$ and $L_2(\mathbb{R}\backslash[-\varepsilon, \varepsilon])$, $\varepsilon > 0$.

Most approaches for non-parametric estimation of the Lévy triplet in the case of Lévy processes are based on Fourier techniques. Instead of using plug-in estimators as above, we can estimate the Fourier transform of $x\nu_1$ directly from the data $X(t_1),\ldots,X(t_l)$ Multiplying both sides of equation (5) with x and taking the Fourier transform $\mathcal{F}$ one gets

$$\mathcal{F}\left[x\nu_0\right](x) = \frac{1}{\nu_d(\Delta_1)}\left(\frac{1}{f_1}\mathcal{F}\left[x\nu_1\right](\frac{1}{f_1}x) + \sum_{k=1}^{\infty}(-1)^k \sum_{i_1,\ldots,i_k=2}^{m} \frac{f_{i_1}\cdots f_{i_k}}{f_1^{k+1}}\mathcal{F}\left[x\nu_1\right](\frac{f_{i_1}\cdots f_{i_k}}{f_1^{k+1}}x)\right)$$

If it is assumed that the series on the right-hand side converges and that the observations $X(t_1),\ldots,X(t_l)$ are independent, then one could use directly the formulae in to estimate the Fourier transform of of $x\nu_1$:

$$\widehat{\mathcal{F}\left[x\nu_1\right]}(x) = (-i)\frac{\hat{\varphi}_l'(x)}{\hat{\varphi}_l(x)}\mathbb{1}_{|\{\hat{\varphi}_l(x)|>l^{-\frac{1}{2}}\}},$$

where φ_l denotes the empirical characteristic function of $X(t_1),\ldots,X(t_l)$.

The resulting estimator $\bar{v}_0$ for the density v_0 then is given by

$$\bar{v}_0(u) = \frac{1}{u \cdot \nu_d(\Delta_1)} \mathcal{F}^{-1} \left[\left(\frac{1}{\hat{f}_1} \widehat{\mathcal{F}[x v_1]}(\frac{1}{\hat{f}_1} x) \right. \right.$$
$$\left. \left. + \sum_{k=1}^{\infty} (-1)^k \sum_{i_1, \ldots, i_k = 2}^{m} \frac{\hat{f}_{i_1} \cdots \hat{f}_{i_k}}{\hat{f}_1^{k+1}} \widehat{\mathcal{F}[x v_1]}(\frac{\hat{f}_{i_1} \cdots \hat{f}_{i_k}}{\hat{f}_1^{k+1}} x) \right) \right] (u)$$

We would like to show the L_1- or L_2 consistency of $\bar{v}_0$ and the rate of convergence to v_0 under the assumption of weakly dependent observations.

Now assume for simplicity that $u^2 v_0(u)$ is integrable on $\mathbb{R}$ and $f(-x) = f(x)$, $x \in \mathbb{R}^d$. Moreover let all natural powers of f be integrable on $\mathbb{R}^d$. Then one can show that the cumulant of $X(t)$ looks like

$$\log \varphi_{X(t)}(z) = ia_0 z \int_{\mathbb{R}^d} f(x)dx - b_0 \frac{z^2}{2} \int_{\mathbb{R}^d} f^2(x)dx - z^2 \int_{\mathbb{R}} \zeta(z,y) y^2 v_0(y)dy,$$

where

$$\zeta(z,y) = \sum_{k=2}^{\infty} \frac{(izy)^{k-2}}{k!} \int_{\mathbb{R}^d} f^k(x)dx, \quad z, y \in \mathbb{R}.$$

The above cumulant function can be approximated by a sequence of functions $g_M : \mathbb{R} \to \mathbb{R}$ as $M \to \infty$ given by

$$g_M(z) = ia_0 z \int_{\mathbb{R}^d} f(x)dx - b_0 \frac{z^2}{2} \int_{\mathbb{R}^d} f^2(x)dx - z^2 \int_{-M}^{M} \zeta(z,y) y^2 v_0(y)dy$$

Fix an orthonormal basis $\{\psi_j\}_{j\in\mathbb{N}}$ in $L_2([-M,M])$ with scalar product $\langle h_1,h_2\rangle_2 = \int_{-M}^{M} h_1(x)h_2(x)dx$, $h_1,h_2\in L_2([-M,M])$. Then it holds

$$g_M(z) = ia_0 z\int\limits_{\mathbb{R}^d} f(x)dx - b_0\frac{z^2}{2}\int\limits_{\mathbb{R}^d} f^2(x)dx - z^2\sum_{j=1}^{\infty}\langle\zeta(z,\cdot),\psi_j\rangle_2\left\langle y^2 v_0(\cdot),\psi_j\right\rangle_2 .$$

Introduce the parameter vector $\beta=(\beta_j)_{j=-1,0,1,2,\ldots}$ given by $\beta_{-1}=a_0,\ \beta_0=b_0$, $\beta_j=\langle\zeta(z,\cdot),\psi_j\rangle_2$, $j\in\mathbb{N}$ as well as the vector-valued function

$$F_z = \left(iz\int\limits_{\mathbb{R}^d} f(x)dx, -\frac{z^2}{2}\int\limits_{\mathbb{R}^d} f^2(x)dx, -z^2\left\{\left\langle y^2 v_0(\cdot),\psi_j\right\rangle_2\right\}_{j\in\mathbb{N}}\right), \quad z\in\mathbb{R}.$$

Then we can formally write

$$g_M(z) = \langle F_z,\beta\rangle$$

as a linear function of $\beta\in l_2$ with coefficients in F_z.

Let $W_l = [0, n_l]^d$ be an observation window, where $n_l \to \infty$ as $l \to \infty$. Assume that a sample $X(t_1), \ldots, X(t_l)$ is given, where $t_1, \ldots, t_l \in W_l$ for any $l \in \mathbb{N}$. Introduce the empirical characteristic function of $X(0)$ by $\hat{\varphi}_{X(0)}(z) = (\nu_d(W_l)^{-1}) \int_{W_l} e^{izX(t)} dt, \; z \in \mathbb{R}$. Due to the stationarity of X it then holds that $E\hat{\varphi}_{X(0)}(z) = \varphi_{X(0)}(z)$ for any $z \in \mathbb{R}$.
The idea now is to use a least squares method to define

$$\hat{\beta}_k^M = \operatorname{argmin}_{\beta \in l_2^k} \left(\sup_{z \in \mathbb{R}} |\langle F_z, \beta \rangle - \log \hat{\varphi}_l(z)|^2 + \lambda \mathrm{Pen}(\beta) \right)$$

where $l_2^k = \{x \in l_2 : x = (x_{-1}, x_0, x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots)\}$ for any $k \in \mathbb{N}$, $\lambda \geq 0$ is a weight parameter and $\mathrm{Pen}(\beta)$ is a penalty function which governs e.g. the smoothness of Lévy densities v_0 we would like to get at the end of the estimation procedure. As a final estimator of the Lévy triplet (a_0, b_0, v_0) we propose

$$\hat{\beta} = \lim_{k \to \infty} \lim_{M \to \infty} \hat{\beta}_k^M.$$

Contents

Open Questions

Investigate

- consistency and robustness
- upper and lower bounds for the estimation error
- asymptotic distribution

of the above estimators and compare their performance.

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Thank you for your attention!