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Rigid analytic curves and their Jacobians

Workshop “Probability, Analysis and Geometry”

Sophie Schmieg | September 2013 | Institute of Pure Mathematics

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Valuations

To be able to do analysis, one needs a field and an absolute value.

Definition

A field K together with $|\cdot|\colon K \to \mathbb{R}_0^+$ is called a *valued field*, if

(i) $|x| = 0$ if and only if $x = 0$.

(ii) $|xy| = |x| \cdot |y|$ for all $x, y \in K$.

(iii) $|x + y| \leq |x| + |y|$ for all $x, y \in K$.

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Usually for $K = \mathbb{Q}$ one defines

$$\left|\frac{a}{b}\right| = \begin{cases} \frac{a}{b} & \text{if } \frac{a}{b} \geq 0 \\ -\frac{a}{b} & \text{if } \frac{a}{b} < 0 \end{cases}$$

Valuations

Let us instead set

$$\left|\frac{a}{b}\right| = \begin{cases} 0 & \text{if } a = 0 \\ p^{\nu(b)-\nu(a)} & \text{else} \end{cases}$$

with

- p prime
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We get the stronger version of (iii)

(iii') $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in K$.

and call the field a *non-Archimedean valued field*

Consequences

- Totally disconnected topology

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$$\sum_{k=0}^{\infty} a_k \text{ converges } \Leftrightarrow a_k \to 0$$

- Hensel's lemma: Newton's method convergence a priori
- Close connection to the finite field $\mathbb{F}_p$

Repairing the topology

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Definition

X set, $\mathfrak{S} \subset \mathcal{P}(X)$ set of subsets of X, $\{\mathrm{Cov}\, U\}_{U \in \mathfrak{S}}$ family of coverings.

(i) $U, V \in \mathfrak{S} \Rightarrow U \cap V \in \mathfrak{S}$.

(ii) $U \in \mathfrak{S} \Rightarrow \{U\} \in \mathrm{Cov}\, U$.

(iii) If $U \in \mathfrak{S}, \{U_i\}_{i \in I} \in \mathrm{Cov}\, U$ and $\{V_{ij}\}_{j \in J_i} \in \mathrm{Cov}\, U_i$, then the covering $\{V_{ij}\}_{i \in I, j \in J_i}$ is also admissible.

(iv) If $U, V \in \mathfrak{S}$ with $U \subset V$ and $\{V_i\}_{i \in I} \in \mathrm{Cov}\, V$, then the covering $\{V_i \cap U\}_{i \in I}$ of U is admissible.

Reduction

If K is a Non-Archimedean valued field, then $R := \{x \in K \,;\, |x| \leq 1\}$ is a ring and $\mathfrak{m} := \{x \in K \,;\, |x| < 1\}$ is a maximal ideal in R, $k := R/\mathfrak{m}$.

Example

$K = \mathbb{Q}_p$, $a_k \in \{0, \ldots, p-1\}, m \in \mathbb{N}_0$,
$x = \sum_{k=-m}^{\infty} a_k p^k, a_{-m} \neq 0$.

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X curve over K, $\tilde{X}$ curve over k.

Divisors

$X\colon y^2 = x(x+1)(x-1)$ elliptic curve

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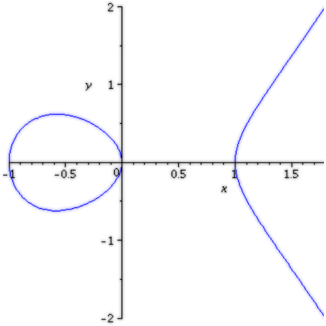

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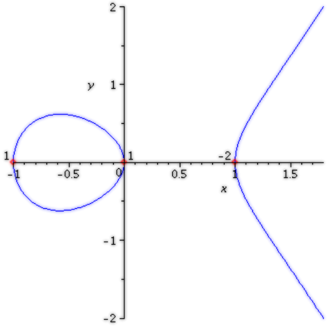
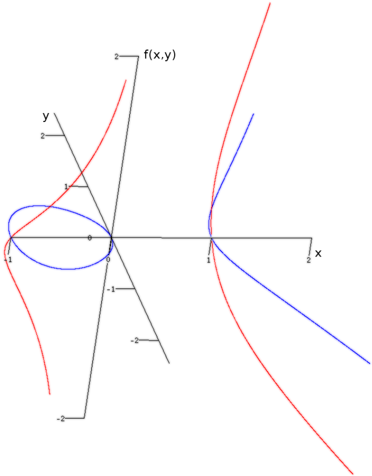


$D = (-1,0) + (0,0) - 2(1,0) \qquad \deg D = 1 + 1 - 2 = 0$

Divisors and principal divisors: Example 1

$$f_1 = \frac{y}{x}$$



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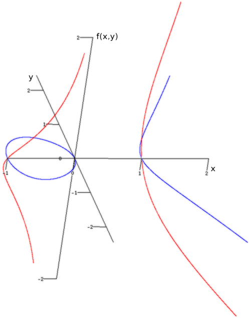
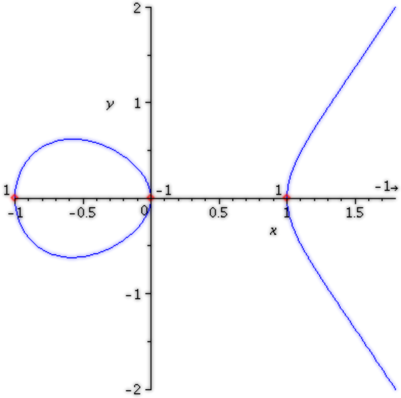
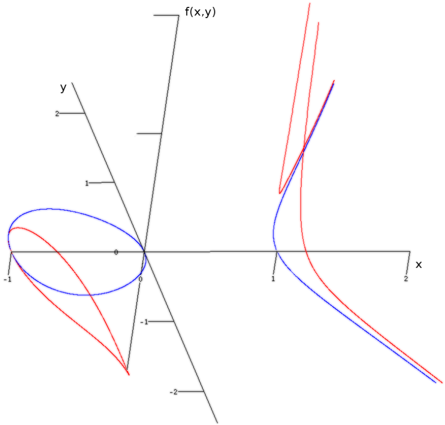




$$f_1 = \frac{y}{x} \qquad \operatorname{div} f_1 = (-1,0) - (0,0) + (1,0) - \infty$$

Divisors and principal divisors: Example 2

$$f_2 = \frac{x+1}{x-1}$$



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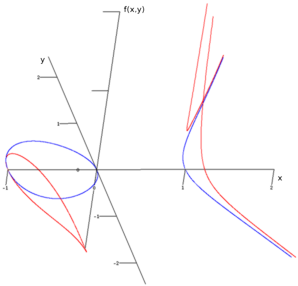
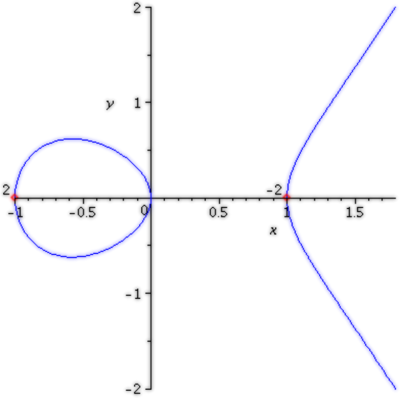




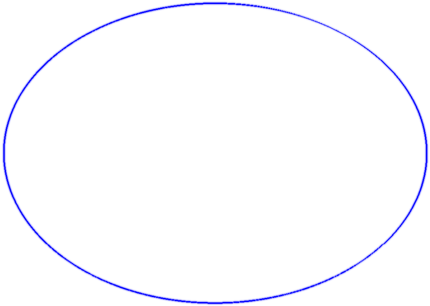
$$f_2 = \frac{x+1}{x-1}$$

$$\operatorname{div} f_2 = 2(-1,0) - 2(0,0)$$

Line bundles

New curve: circle parametrized by φ.

$$X: \{(\cos\varphi, \sin\varphi)\ ;\ \varphi \in [-\pi, \pi]\}$$



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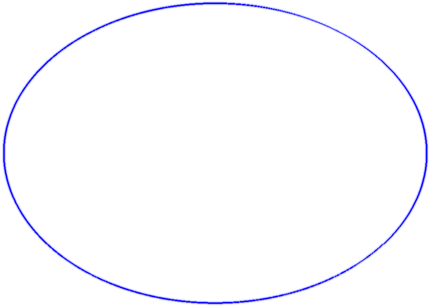
What is meant by saying a scalar field is continuous on X?



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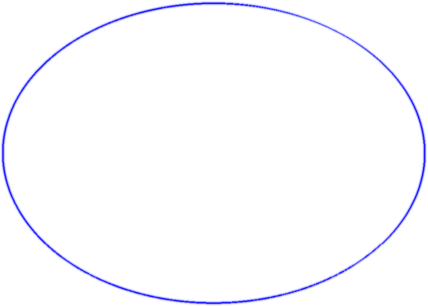
What is meant by saying a scalar field is continuous on X?

$$f_1 = \cos(5\varphi) \cdot \ell$$

$$f_2 = \cos\left(\frac{11}{2}\varphi\right) \cdot \ell$$



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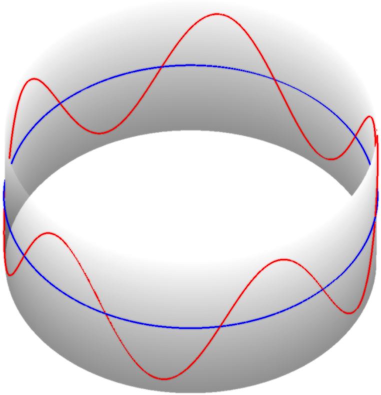

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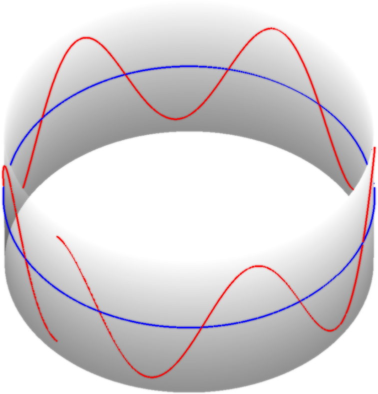

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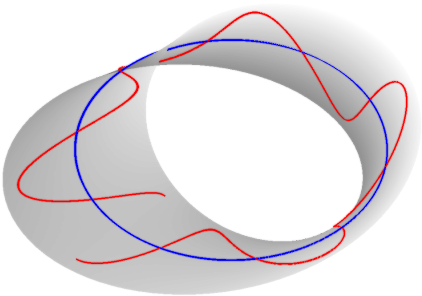

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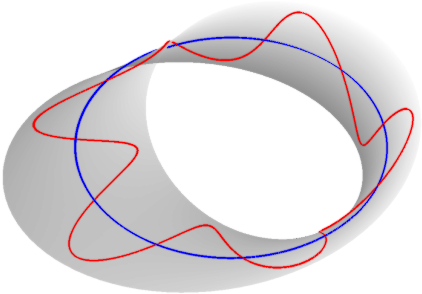

Line bundles: Notations

One denotes

- $f_1 \in H^0(X, \mathcal{O}_X)$, f_1 is a global section of the trivial line bundle $\mathcal{O}_X$, the bundle of functions on X.
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Important example

Ω^1_X, the line bundle of holomorphic differential forms of a Riemann surface X.
The “unit” here is $\mathrm{d}z$ with z local parameter on X.

Jacobian variety

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(iii) Invertible sheaves

(iv) $H^1(X, \mathcal{O}_X^\times)$.

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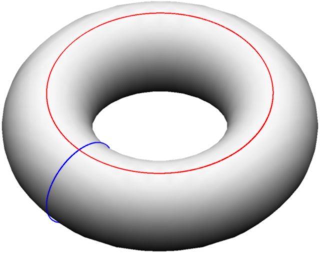
This object is a *commutative group*.
Restriction to divisors of degree 0 is called the *Jacobian variety of X*.

Topology of Riemann surfaces

X compact Riemann surface of genus g.

(i) $\pi_1(X) = \langle a_i, b_i | a_1 b_1 \dots a_g b_g a_1^{-1} b_1^{-1} \dots a_g^{-1} b_g^{-1} \rangle$.



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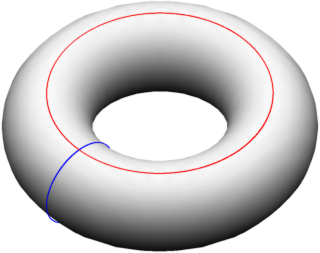

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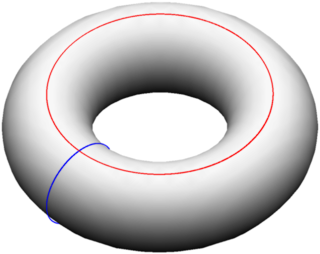
(iii) $\dim_{\mathbb{C}} H^0(X, \Omega) = g$.



Topology of Riemann surfaces

Canonical pairing

$$\begin{aligned} H_1(X,\mathbb{Z}) \times H^0(X,\Omega) &\to \mathbb{C} \\ (\gamma,\omega) &\mapsto \int_\gamma \omega \quad . \end{aligned}$$

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- M lattice in $\mathbb{C}^g$, composed of every possible value of integrals of ω_j over closed curves.

Theorems of Abel and Jacobi

Theorem

$$\operatorname{Div}^0 X \to \mathbb{C}^g/M$$
$$\sum_{i\in I}[x_i - y_i] \mapsto \left(\sum_{i\in I}\int_{y_i}^{x_i}\omega_j\right)_{j=1}^{g}$$

is surjective and its kernel are the principal divisors.

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$\operatorname{Jac} X$ has a holomorphic structure.

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We have a group homomorphism in $\mathbb{C}$:

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$\operatorname{Jac} X \cong \mathbb{G}_{m,\mathbb{C}}^{g} / \exp(2\pi iM)$,
$\exp(2\pi iM)$ is a multiplicative lattice of rang g.

Semi stable reduction

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There is a formal covering $\mathfrak{U}$ such that the associated reduction has only ordinary double points as singularities.

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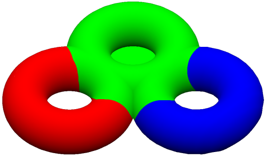
Definition

$\tilde{X}$ semi-stable, *dual graph G*:

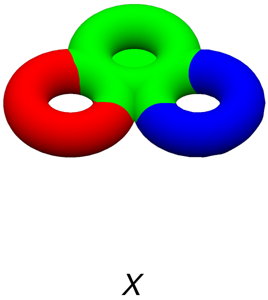
$$V(G) = \text{irreducible components} \qquad \text{vertex set}$$
$$E(G) = \text{double points} \qquad \text{edge set}$$

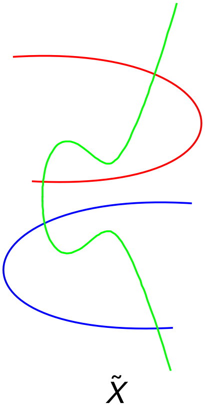
Example



X

Example





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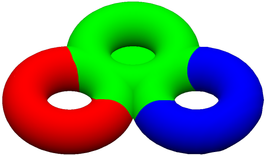

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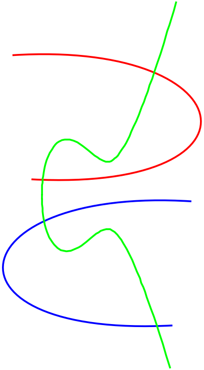
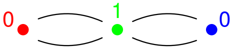


$\tilde{X}$



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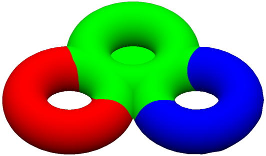

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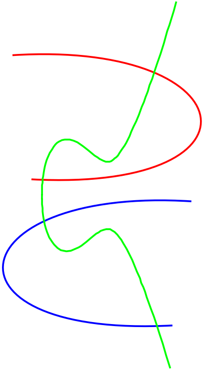

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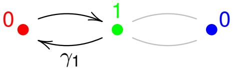

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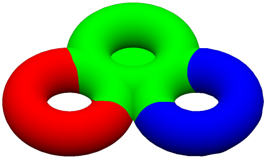

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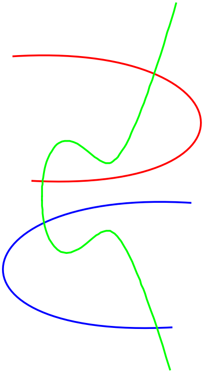

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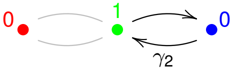
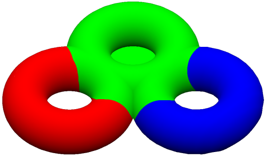
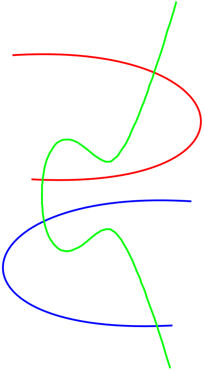


G

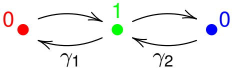
Example



X



$\tilde{X}$



G

The rigid analytic case

Let X be a rigid analytic curve over K. There is an abelian variety B over K and an extension

$$0 \to \mathbb{G}_{m,K}^{t} \to \hat{J} \to B \to 0$$

and a lattice M in $\hat{J}$ such that

$$\operatorname{Jac} X = \hat{J}/M$$

The lattice

The lattice $-\log|M| \subset \mathbb{R}^t$ has the base (v_i) with

$$v_{ij} = \sum_{e \in \gamma_i \cap \gamma_j} -d(e) \cdot \log|q(e)| \ ,$$

where

- γ_i the simple cycles of G,
- $d(e) = 1$ if γ_i and γ_j have the same direction in e, $d(e) = -1$ otherwise
- $q(e)$ the height of the annulus corresponding to e .