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Optimal stopping problems for Brownian motion with drift and disorder;

application to

mathematical finance and engineering

§ 1. ESTIMATION of the DRIFT of BROWNIAN MOTION

INTRODUCTION

We consider two models of observed processes $(X_t)_{t\geq 0}$ driven by **Brownian motion** $(B_t)_{t\geq 0}$.

Model A: *(Part I)*

$$\boxed{X_t = \mu t + B_t} \qquad \text{or, in differentials, } dX_t = \mu\, dt + dB_t,$$

where μ is a **random parameter** which does not depend on B.

Model B: *(Part II)*

$$\boxed{X_t = \mu(t-\theta)^+ + B_t} \qquad \text{or} \quad dX_t = \begin{cases} dB_t, & t < \theta, \\ \mu\, dt + dB_t, & t \geq \theta, \end{cases}$$

where (μ, θ) are **random parameters** which do not depend on B.

Our presentation are based on the recent works:

- *U. Cetin, A. A. Novikov, A. Shiryaev.* A Bayesian estimation of drift of fractional Brownian motion
  (Preprints, LSE, UTS.)

- *A. Shiryaev, M. Zhitlukhin.* A Bayesian sequential testing problem of three hypotheses for Brownian motion.
  (Statistics & Risk Modeling, 2011, No. 3)

- *M. Zhitlukhin, A. Shiryaev.* Bayesian disorder problems on filtered probability spaces
  (TPA, 2012, No. 3)

- *A. Aliev* Towards a problem of detection of a disorder which depends on trajectories of the process **(TPA, 2012, No. 3)**

- *M. Zhitlukhin, A. Muravlev.* Solution of a Chernoff problem of testing hypotheses on drift of Brownian motion **(TPA, 2012, No. 4)**

- *A. Shiryaev, M. Zhitlukhin.* Optimal stopping problems for a Brownian motion with a disorder on a finite interval **(TPA, 2013)**

We consider some problems of financial economics which can be solved by the methods of optimal stopping. The general problem of such type can be formulated as follows: To find the value function

$$V(T) = \sup_{\tau \le T} \mathsf{E} G_\tau,$$

where τ is a stopping time, T is a finite horizon. Of course, it is interesting to find also the optimal stopping time τ^* for which $\mathsf{E} G_{\tau^*} = V(T)$ (if this stopping time exists).

A lot of books are written on optimal stopping. For example,

G. Peskir and A. Shiryaev.

Optimal stopping and free-boundary problems.

We would like to expose here our results obtained together with several our colleagues (A. Novikov, X. Y. Zhou,...).

ESTIMATION of the DRIFT COEFFICIENT

We observe a process $X = (X_t)_{t \geq 0}$

$$\boxed{X_t = \mu t + B_t}$$

where μ is a **random parameter** which does not depend on B.

Decision rule based on $\mathcal{F}^X$-observations ($\mathcal{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$, $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$), is a pair $\delta = (\tau, d)$, where

- τ is a $\mathcal{F}^X$-stopping time (i.e., $\{\tau \leq t\} \in \mathcal{F}_t^X$ for any $t \geq 0$);
- d is a $\mathcal{F}_\tau^X$-measurable function (taking values in $\mathbb{R}$).

The **Bayesian risk** which we consider is given by

$$\mathcal{R} = \inf_{(\tau,d)} \mathsf{E}[c\tau + W(\mu, d)],$$

where

- E is the mean with respect to the measure generated by (independent) μ and B;
- W is a **penalty function**; $\mathsf{E}\tau < \infty$.

Due to the representation

$$\mathsf{E}[c\tau + W(\mu, d)] = \mathsf{E}\left\{\mathsf{E}\left[c\tau + W(\mu, d) \,\middle|\, \mathcal{F}_\tau^X\right]\right\}$$

and the $\mathcal{F}_\tau^X$-measurability of τ and d, we need to find

$$\mathsf{E}\left[W(\mu, d) \,\middle|\, \mathcal{F}_\tau^X\right].$$

The conditional distribution of μ is determined by

$$\mathsf{P}\Big(\mu \le y \,\Big|\, \mathcal{F}_t^X\Big) = \frac{\displaystyle\int_{-\infty}^{y} \frac{d\mathsf{P}(X_0^t \,|\, \mu = z)}{d\mathsf{P}(X_0^t \,|\, \mu = 0)}\, dP_\mu(z)}{\displaystyle\int_{-\infty}^{\infty} \frac{d\mathsf{P}(X_0^t \,|\, \mu = z)}{d\mathsf{P}(X_0^t \,|\, \mu = 0)}\, dP_\mu(z)},$$

with the **Radon–Nikodým derivative**

$$\frac{d\mathsf{P}(X_0^t \,|\, \mu = z)}{d\mathsf{P}(X_0^t \,|\, \mu = 0)}$$

of the measure of the process $X_0^t = (X_s, s \le t)$ with $\mu = z$ w.r.t.
the measure of the process $X_0^t = (X_s, s \le t)$ with $\mu = 0$.

Calculating explicitly the Radon–Nykodým derivative, we find

$$\mathsf{P}\Big(\mu \le y \,\Big|\, \mathcal{F}_t^X\Big) = \frac{\int_{-\infty}^{y} e^{zX_t - z^2 t/2}\, dP_\mu(z)}{\int_{-\infty}^{\infty} e^{zX_t - z^2 t/2}\, dP_\mu(z)}.$$

If $P_\mu(z)$ has a density, $dP_\mu(z) = p(z)dz$,
then the **conditional density μ** admits the representation

$$p(y, X_t; t) := \frac{d\mathsf{P}(\mu \le y \,|\, \mathcal{F}_t^X)}{dy} = \frac{e^{yX_t - y^2 t/2} p(y)}{\int_{-\infty}^{\infty} e^{zX_t - z^2 t/2} p(z)\, dz}.$$

Thus, for $d = d(\tau)$ we have

$$\mathsf{E}[W(\mu, d) \,|\, \mathcal{F}_\tau^X] = \int_{\mathbb{R}} W(y, d(\tau)) \cdot p(y, X_\tau, \tau)\, dy.$$

If for each τ there exists an $\mathcal{F}_\tau^X$-measurable function $d^*(\tau)$ such that

$$\inf_{d \in \mathcal{F}_\tau^X} \int_{\mathbb{R}} W(y, d) \cdot p(y, X_\tau; \tau) dy =$$

$$= \int_{\mathbb{R}} W(y, d^*(\tau)) \cdot p(y, X_\tau; \tau) dy \qquad (\equiv G(\tau, X_\tau))\,,$$

then (with the notation $p = \mathsf{Law}\, \mu$)

$$\inf_{(\tau,d)} \mathsf{E}[c\tau + W(\mu, d)] = \inf_\tau \mathsf{E}[c\tau + G(\tau, X_\tau)] \qquad (\equiv V(p)).$$

If τ^* is an **optimal time** for the right-hand side,
then $(\tau^*, d^*(\tau^*))$ is an **optimal solution** of the initial problem.

EXAMPLE 1 (classical mean-square criterion)

$$W(\mu, d) = (\mu - d)^2 \quad \text{and} \quad \mu \sim \mathcal{N}(m, \sigma^2)$$

In this case

$$V(p) = \inf_{\tau} \mathsf{E}[c\tau + v(\tau)], \qquad \text{where} \quad v(t) = 1/(t + \sigma^{-2}).$$

The optimal time τ^* is **deterministic**, at that

(a) if $\sqrt{c} < \sigma^2$, then τ^* is a unique solution to the equation $v(\tau^*) = \sqrt{c}$, i.e., $\tau^* = c^{-1/2} - \sigma^{-2}$;

(b) if $\sqrt{c} \geq \sigma^2$, then $\tau^* = 0$.

Optimal d^* coincides with the **a posteriori mean** $\mathsf{E}(\mu \,|\, \mathcal{F}^X_{\tau^*})$:

(c) $$d^* = \begin{cases} \sqrt{c} X_{\tau^*} + m\sqrt{c}/\sigma^2, & \text{if } \sqrt{c} < \sigma^2, \\ m, & \text{if } \sqrt{c} \geq \sigma^2. \end{cases}$$

How can one get the representation

$$V(p) = \inf_{\tau} \mathsf{E}[c\tau + v(\tau)] \qquad \text{for} \qquad v(t) = 1/(t+\sigma^{-2}) \ \ \textbf{?}$$

Consider

$$\inf_{(\tau,d)} \mathsf{E}[c\tau + (\mu - d)^2].$$

For a given τ the **optimal** $d^*(\tau)$ is $\mathsf{E}(\mu \,|\, \mathcal{F}_\tau^X)$:

$$\boxed{d^*(\tau) = \int_{\mathbb{R}} y \cdot p(y, X_\tau; \tau)\, dy.}$$

It is interesting to observe that if we denote

$$A(t,x) = \int_{\mathbb{R}} y \cdot p(y,x;t)\,dy,$$

then from the explicit form of $p(y,x;t)$ we can see that

$$A'_x(t,x) = \int_{\mathbb{R}} y^2 \cdot p(y,x;t)\,dy - A^2(t,x).$$

So, $A'(t,X_t) = \mathsf{E}\left[(\mu - \mathsf{E}(\mu \,|\, \mathcal{F}_t^X))^2 \,\middle|\, \mathcal{F}_t^X\right]$. Thus,

> $A'_x(t,X_t)$ is the **variance of** μ conditioned on $\mathcal{F}_t^X$.

Consequently,

$$V(p) = \inf_\tau \mathsf{E}[c\tau + A'_x(\tau, X_\tau)] \qquad \left(\equiv \inf_\tau \mathsf{E}[c\tau + G(\tau, X_\tau)]\right).$$

If $\mu \sim \mathcal{N}(m, \sigma^2)$, then the **conditional variance** has the form

$$\boxed{A_x(t, X_t) = v(t),}$$

where $v(t)$ solves the Riccati equation (**Kalman–Bucy filter**)

$$v'(t) = -v^2(t), \quad v(0) = \sigma^2,$$

i. e.,

$$v(t) = \frac{1}{t + \sigma^{-2}}.$$

Thus,

$$V(p) = \inf_{\tau} \mathsf{E}\left[c\tau + \frac{1}{t + \sigma^{-2}}\right],$$

which proves **(a)** and **(b)** for τ^*.

Representation **(c)** for $d^* = \mathsf{E}(\mu \,|\, \mathcal{F}^X_{\tau^*})$ follows from the formula

$$\begin{aligned} d^*(\tau^*) &= \int_{\mathbb{R}} y p(y, X_{\tau^*}; \tau^*)\, dy \\ &= X_{\tau^*} v(\tau^*) + m \exp\left(- \int_0^{\tau^*} v(s)\, ds \right) \\ &= X_{\tau^*} \frac{\sigma^2}{1+\sigma^2 \tau^*} + \frac{m}{1+\sigma^2\tau^*}, \end{aligned}$$

whence we find

$$d^*(\tau^*) = \begin{cases} \sqrt{c} X_{\tau^*} + m\sqrt{c}/\sigma^2, & \text{if } \sqrt{c} < \sigma^2 \ \ (\tau^* = c^{-1/2} - \sigma^{-2}), \\ m, & \text{if } \sqrt{c} \ge \sigma^2 \ \ (\tau^* = 0). \end{cases}$$

EXAMPLE 2 (criterion connected with the precise detection, when $d^* = \mu$)

$$\boxed{W(\mu, \cdot) = -\epsilon_\mu(\cdot)}$$

where ϵ_μ is a Dirac function. In this case

$$\int_{\mathbb{R}} W(\mu, d) p(\tau, X_\tau, y)\, dy = -p(\tau, X_\tau, d) = -\frac{p(d) \exp(X_\tau d - \frac{1}{2}\tau d^2)}{\int_{\mathbb{R}} p(z) \exp(xz - \frac{1}{2}\tau z^2)\, dz}.$$

Thus, $d^*(\tau)$ is a **mode** of the conditional density $p(\tau, X_\tau, \cdot)$ (i.e., any point of **local maximum** $p(\tau, X_\tau, \cdot)$).

If the support of p is $\mathbb{R}$ and the function p is differentiable, then $d^*(\tau)$ solves the equation

$$\frac{p'(d)}{p(d)} - \tau d = -X_\tau.$$

In **normal case** $\mu \sim \mathcal{N}(m, \sigma^2)$ the **mode** coincides with the **conditional mean** (see **Example 1**):

$$d^*(\tau) = \begin{cases} \sqrt{c}X_\tau + m\sqrt{c}/\sigma^2, & \text{if } \sqrt{c} < \sigma^2 \ \ (\tau = c^{-1/2} - \sigma^{-2}), \\ m, & \text{if } \sqrt{c} \geq \sigma^2 \ \ (\tau = 0). \end{cases}$$

In this case

$$G(\tau, X_\tau) = -p(\tau, X_\tau; d^*(\tau)) = -\frac{1}{\sqrt{2\pi v(\tau)}}\,.$$

Taking into account that $\mathsf{E}(c\tau + G(\tau, X_\tau)) = \mathsf{E}(c\tau - 1/\sqrt{2\pi v(\tau)})$, we obtain the equality that $\tau^* = t^*$, where

$$c - \frac{1}{2}\sqrt{\frac{v(t^*)}{2\pi}} = 0.$$

Consequently,

$$t^* = \begin{cases} 1/(8\pi c^2) - 1/\sigma^2, & \text{if } 8\pi c^2 < \sigma^2, \\ 0, & \text{if } 8\pi c^2 \geq \sigma^2. \end{cases}$$

The corresponding function d^* is given by

$$d^* = v(\tau^*) X_{\tau^*} + m\frac{v(\tau^*)}{\sigma^2} = 8\pi c^2 X_{\tau^*} + m\frac{8\pi c^2}{\sigma^2}.$$

Of great interest are problems, where μ lies in a **finite interval** $[\mu_1, \mu_2]$ with, e.g., **uniform distribution**. In this case optimal time τ^* is **NOT deterministic**.

§ 2. Bayesian sequential estimation of the drift of fractional Brownian motion

We assume the observed process $X = (X_t)_{t \geq 0}$ has the representation

$$X_t = \theta t + B_t^H,$$

where $B^H = (B_t^H)_{t \geq 0}$ is a fractional Brownian motion with

$$B_0^H = 0, \quad \mathsf{E} B_t^H = 0, \quad \mathsf{E}|B_t^H - B_s^H|^2 = |t - s|^{2H}, \quad 0 < H < 1.$$

In case $H \neq 1/2$ the process B^H is not a semimartingale; in case $H = 1/2$ the process $B^{1/2}$ is a Brownian motion.

We consider the problem:

To find a sequential optimal rule $\delta^* = (\tau^*, d^*)$

$$\inf_{\delta \in D} \mathsf{E}[c\tau + w(\theta, d(\tau))] = \mathsf{E}[c\tau^* + w(\theta, d^*)],$$

where D is a class of rules with stopping time $\tau \le T < \infty$ w.r.t. the flow $\mathcal{F}_t^X = \sigma(X_s;\ s \le t)$; $d(\tau)$ is $\mathcal{F}_\tau^X$-measurable; $w(\tau, d)$ is a "penalty" function.

In this talk we consider the penalty functions

$$w(\theta, d) = |\theta - d|^2 \quad \text{and} \quad w(\theta, d) = -\delta(\theta, d),$$

where $\delta(\theta, d)$ if the Dirac delta function which can be understood as the distributional limit as $\varepsilon \to 0$:

$$\delta(\theta, d; \varepsilon) = \begin{cases} -1/(2\varepsilon) & \text{if } d \in (\theta - \varepsilon, \theta + \varepsilon), \\ 0 & \text{if } d \notin (\theta - \varepsilon, \theta + \varepsilon). \end{cases}$$

If $\mathsf{E}|w(\theta,d)|<\infty$, then $\mathsf{E}(c\tau+w(\theta,d(\tau)))=\mathsf{E}[c\tau+\mathsf{E}(w(\theta,d(\tau))\,|\,\mathcal{F}_\tau^X)]$. By the generalized Bayes formula, the conditional density

$$p(y;X,t)=\frac{d\mathsf{P}(\theta\le y\,|\,\mathcal{F}_t^X)}{dy}$$

has the following representation:

$$p(y;X,t)=\frac{p(y)L_t(y,X)}{\int_E p(y)L_t(y,X)\,dy},$$

where $p(y)$, $y\in E$, is a density of the distribution of θ and $L_t(y,X)$ is a Radon–Nikodým derivative of the measure generated by $X_u=yu+B_u^H$ (on $(\Omega,\mathcal{F},(\mathcal{F}_t^X)_{t\ge0},\mathsf{P})$) w.r.t. the measure of $X_u=B_u^H$, $u\le t$.

LEMMA 1 (**Norros, Valkeila, Virtano** [`Bernoulli 5 (1999), 571-587`]). We have

$$L_t(y, B^H) = \exp\Big\{ y M_t(B^H) - \frac{y^2}{2} \langle M(B^H) \rangle_t \Big\},$$

where $M = (M_t(B^H))_{t \geq 0}$ is a fundamental Gaussian martingale with independent increments such that

$$\langle M \rangle_t = \mathsf{D}(M_t(B^H)) = C_2^2 t^{2-2H},$$

$$C_2^2 = \frac{\Gamma(3/2 - H)}{4H(1-H)\Gamma(1/4 + H)\Gamma(2-2H)},$$

$$M_t(B^H) = c \int_0^\infty K(t,s)\, dB_s^H, \quad K(t,s) = C_1 (st - s^2)^{1/2 - H}, \quad s \in (0,t),$$

$$C_1 = 2HB(3/2 - H, 1/2 + H)^{-1},$$

where $B(x,y)$ is a beta function (Euler integral of the first kind).

So,

$$p(y; X, t) = \frac{p(y) \exp\{yM_t(X) - \frac{y^2}{2}\langle M \rangle_t\}}{\int p(y) \exp\{yM_t(X) - \frac{y^2}{2}\langle M \rangle_t\}\, dy},$$

and hence the optimal d^* should be found from the following relation:

$$\inf_d \mathsf{E}[w(\theta, d) \,|\, \mathcal{F}_\tau^X] \equiv \mathsf{E}[w(\theta, d^*) \,|\, \mathcal{F}_\tau^X] = \int w(y, d^*) p(y; X, \tau)\, dy.$$

The case of **QUADRATIC PENALTY FUNCTION**

Here we consider the case $w(\theta, d) = |\theta - d|^2$. It is well known that $\inf \mathsf{E}(|\mu - d(\tau)|^2 \,|\, \mathcal{F}_\tau^X)$ is achieved for $\tau < \infty$ with the decision function

$$d^*(\tau) = \mathsf{E}(\mu \,|\, \mathcal{F}_\tau^X) = \int y p(y; X, \tau)\, dy.$$

LEMMA 2. Let $\theta \sim N(m, 1)$. Then for any $t \geq 0$

$$\mathsf{E}(\theta \,|\, \mathcal{F}_t^X) = \frac{m + M_t(X)}{1 + \langle M \rangle_t} \quad \text{and} \quad \mathsf{D}(\theta \,|\, \mathcal{F}_t) = \frac{1}{1 + \langle M \rangle_t}.$$

Direct calculations show that

$$\inf_{\delta \in D} \mathsf{E}\Big[c\tau + |\theta - d(\tau)|^2\Big] = \inf_{\tau} \mathsf{E}\Big[c\tau + \frac{1}{1 + \langle M \rangle_\tau}\Big] = \inf_{t \in [0,T]} F_H(t),$$

$$\text{where} \quad F_H(t) = ct + \frac{1}{1 + C_2^2 t^{2-2H}}.$$

THEOREM 1. Let $\theta \sim N(m, 1)$, $w(\theta, d) = |\theta - d|^2$. In this case the optimal stopping time τ^* is deterministic and has the form:

1) if $H > 1/2$, then

$$\tau^* = \arg \inf_{t \in [0,T]} F_H(t) = \begin{cases} t_1^*, & \text{if } t_1^* < T, \\ T, & \text{if } t_1^* \geq T, \end{cases}$$

where t_1^* is a solution of the equation

$$cF_H'(t) - \frac{2(1-H)C_2^2 t^{1-2H}}{1 + C_2^2 t^{2-2H}} = 0;$$

2) if $H = 1/2$, then

$$\tau^* = \arg \inf_{t \in [0,T]} F_{1/2}(t) = \begin{cases} 0, & \text{if } c \geq 1, \\ c^{-1/2} - 1, & \text{if } c < 1 \text{ and } T > c^{-1/2} - 1, \\ T, & \text{if } c < 1 \text{ and } T \leq c^{-1/2} - 1; \end{cases}$$

here $F_{1/2}(t) = ct + 1/(1+t)$;

3) if $H \in (0, 1/2)$, then one can easily find that the function $F_H(t)$ has a maximum at the point t_1 and minimum at the point $t_2 > t_1$; then the optimal time τ^* is defined by the relation

$$\tau^* = \begin{cases} 0, & \text{if } T < t_1 \text{ or } t_1 < T < t_2 \text{ and } F(T) \geq 1, \\ t_2, & \text{if } T \geq t_2 \text{ and } F(t_2) < 1, \\ T, & \text{if } t_1 < T < t_2 \text{ and } F(T) \leq 1. \end{cases}$$

The optimal decision function is

$$d^* = \frac{m + M_{\tau^*}(X)}{1 + C_2^2 (\tau^*)^{2-2H}}.$$

The case of the **Dirac-type penalty function** $w(\theta, d) = -\delta(\theta, d)$.

In this case

$$\mathsf{E}[w(\theta, d) \,|\, \mathcal{F}_t^X] = \int_E w(y, d) p(y; X, t)\, dy = -p(d; X, t).$$

So, $\inf \mathsf{E}(w(\theta, d) \,|\, \mathcal{F}_t^X)$ is achieved when d is a **MODE** of the a posteriori density.

Assume further that $p(y)$ is a differentiable function. Then the optimal decision d^* should be a root of the equation

$$\frac{d}{dy} p(d; X, t) = p'(y) e^{y M_t(X) - \frac{y^2}{2}\langle M \rangle_t} + p(y)[M_t(X) - y\langle M \rangle_t] e^{y M_t(X) - \frac{y^2}{2}\langle M \rangle_t} = 0,$$

or, equivalently, $\boxed{\dfrac{p'(y)}{p(y)} + M_t(X) - y\langle M \rangle_t = 0}$.

Assume $\theta \sim N(m,1)$. Then $p'(y)/p(y) = -(y-m)$; hence the optimal decision d^* at any stopping time τ satisfies $-(d^*-m) + M_\tau(X) - d^*\langle M\rangle_\tau = 0$, thus $d^* = \frac{m+M_\tau(X)}{1+\langle M\rangle_\tau}$. Note that the optimal decision d^* is the same as for quadratic penalty function, but the value of the penalty function is different: direct calculations show that

$$\mathsf{E}[w(\theta,d^*)\,|\,\mathcal{F}_\tau^X] = -p(d^*,X,\tau) = -\frac{p(d^*)e^{d^*M_\tau(X)-(d^*)^2\langle M\rangle_\tau}}{\int_{-\infty}^{\infty} p(y)e^{yM_t(X)-\frac{y^2}{2}\langle M\rangle_t}\,dy}$$

$$= -\frac{\sqrt{1+\langle M\rangle_\tau}}{\sqrt{2\pi}}.$$

Thus

$$\inf_{\delta\in D} \mathsf{E}[c\tau + w(\theta,d)] = \inf_{\tau\le T} \mathsf{E}\left(c\tau - \frac{\sqrt{1+\langle M\rangle_\tau}}{\sqrt{2\pi}}\right) = \inf_{t\le T} G_H(t),$$

$$\text{where} \quad G_H(t) = ct - \frac{\sqrt{1+C_2^2 t^{2-2H}}}{\sqrt{2\pi}}.$$

If $H > 1/2$, then the unique minimum of the function $G_H(t)$ on $(0, \infty)$ is achieved at the point s^*, which is a positive root of the equation

$$G'_H(t) = c - \frac{C_2^2(2-2H)t^{1-2H}}{\sqrt{8\pi}\sqrt{1 + C_2^2 t^{2-2H}}} = 0.$$

Hence the optimal time τ^* is defined by the relations

$$\tau^* = \begin{cases} s_1^*, & \text{if } s_1^* < T, \\ T, & \text{if } s_1^* \geq T. \end{cases}$$

If $H = 1/2$, then $G_{1/2}(t) = ct - \sqrt{1 + t)}/\sqrt{2\pi}$, hence the optimal τ^* is defined by the relations

$$\tau^* = \begin{cases} 0, & \text{if } c \geq \dfrac{1}{\sqrt{8\pi}}, \\ \frac{1}{8\pi c^2} - 1, & \text{if } c < \dfrac{1}{\sqrt{8\pi}} \text{ and } T > \dfrac{1}{8\pi c^2} - 1, \\ T, & \text{if } c < \dfrac{1}{\sqrt{8\pi}} \text{ and } T \leq \dfrac{1}{8\pi c^2} - 1. \end{cases}$$

If $H \in (0, 1/2)$, then one can easily find that the function $G_H(t)$ has a maximum at the point s_1^* and then a minimum at the point $s_2^* > s_1^*$. Hence the optimal observation time τ^* is defined by the relations

$$\tau^* = \begin{cases} 0, & \text{if } T < s_1^* \text{ or } s_1^* < T < s_2^* \text{ and } G_H(T) \geq 1, \\ s_2^*, & \text{if } T \geq s_1^* \text{ and } G_H(s_2^*) < 1, \\ T, & \text{if } s_1^* < T < s_2^* \text{ and } G_H(T) \leq 1. \end{cases}$$

Considerations presented above provides the proof of the following theorem.

THEOREM 2. Let $\theta \sim N(m, 1)$ and $w(\theta, d) = -\delta(\theta, d)$. Then the optimal stopping time τ^* is deterministic and has the form given above. The optimal decision function is

$$d^* = \frac{m + M_{\tau^*}(X)}{1 + C_2^2(\tau^*)^{2-2H}}.$$

CONCLUDING REMARK.

Suppose $X_t = \theta \int_0^t f(X,s)\,ds + W_t$ and $w(\theta,d) = (\theta - d)^2$. Here

$$L_t(y,X) = \exp\Big\{ y \int_0^t f(X,s)\,dX_s - \frac{y^2}{2} \int_0^t f^2(X,s)\,ds \Big\}$$

(if, for example, $\mathsf{E}\exp\Big\{\dfrac{y^2}{2}\displaystyle\int_0^T f^2(W,s)\,ds\Big\} < \infty$). Thus we obtain that for any stopping time $\tau < \infty$

$$d_\tau^* = \frac{m + \int_0^\tau f(X,s)\,dX_s}{1 + \int_0^\tau f^2(X,s)\,ds}.$$

It is easy to find that here

$$\inf_{\delta} \mathsf{E}\Big(c\int_0^\tau f^2(X,s)\,ds + (\theta-d)^2\Big) = \inf_{\tau} \mathsf{E}\Big(c\int_0^\tau f^2(X,s)\,ds + \frac{1}{1+\int_0^\tau f^2(X,s)\,ds}\Big).$$

Assume $\int_0^\infty f^2(X,s)\,ds < \infty$. Then the optimal stopping time is

$$\tau^* = \inf\Big\{t \ge 0\colon \int_0^t f^2(X,s)\,ds = t^*(c)\Big\},$$

$$\text{where}\quad t^*(c) = \begin{cases} 0, & \text{if } c \ge 1, \\ c^{-1/2}-1, & \text{if } c < 1. \end{cases}$$

INTERESTING PROBLEM:

Let $\boxed{dX_t = \theta\, dt + dB_t}$, where $(B_t)_{t\geq 0}$ is a Brownian motion. Here

$$\frac{X_t}{t} = \theta + \frac{W_t}{t} \quad \text{and} \quad \frac{W_t}{t} \to 0 \text{ (P-a.s.)}, \quad t \to \infty.$$

So, it is interesting to find $\mathsf{E}_0 \inf\{t\colon |W_s/s| \leq \varepsilon,\ s \geq t\} =: \mathsf{E}_0 \sigma_\varepsilon(\omega)$.

Since for each θ, $-\infty < \theta < \infty$, we have

$$\mathsf{P}_\theta\Big(\sigma_\varepsilon(\omega) \leq \frac{x}{\varepsilon^2}\Big) = \mathsf{P}\Big(\sup_{0\leq t\leq 1} |W_t| < \sqrt{x}\Big),$$

it follows that $\boxed{\mathsf{E}_0 \sigma_\varepsilon(\omega) = \dfrac{c}{\varepsilon^2} \quad \text{for some constant } c}$.

> PROBLEM: To find $\mathsf{E}_0 \sigma_\varepsilon(\omega)$ for the model
>
> $$dX_t = \theta\, dt + dB_t^H,$$
>
> where $(B_t^H)_{t\geq 0}$ is a fractional Brownian motion.

§ 3. Chernoff's problem

We observe a random process

$$X_t = \mu t + B_t,$$

where $\mu \sim \mathcal{N}(\mu_0, \sigma_0^2)$ does not depend on B.

Bayesian risk:

$$\boxed{\mathcal{R}(\tau, d) = \mathsf{E}[c\tau + k|\mu|\,\mathbb{I}\{d \neq \operatorname{sgn}(\mu)\}]}$$

where d is a $\mathcal{F}_\tau^X$-measurable function taking values ± 1:

if $d = +1$, then we accept the hypothesis $H^+\colon \mu > 0$

if $d = -1$, then we accept the hypothesis $H^-\colon \mu \leq 0$.

Quantities $c, k > 0$ are given constants.

In a remarkable way, the Chernoff problem reduces to a **problem on optimal stopping of the absolute value of Wiener process**.

For fixed μ_0 and σ_0^2, introduce a process $W = (W_t)_{t \le 1}$,

$$\boxed{W_t = \sigma_0(1-t)X_{t/\sigma_0^2(1-t)} - t\mu_0/\sigma_0}$$

where W_1 is defined as the limit of W_t as $t \to 1$.

One can prove that W is a **Wiener process**,
$\mathsf{E}W_t = 0$, $\mathsf{E}W_t^2 = t$ and $W_0 = 0$.

The theorem below shows that to find an optimal decision rule in the initial problem

$$\inf_{(\tau,d)} \mathcal{R}(\tau,d) = \inf_{\tau,d} \mathsf{E}[c\tau + k|\mu|\,\mathbb{I}\{d \neq \operatorname{sgn}(\mu)\}] \qquad \textbf{(A)}$$

it suffices to find

$$V_{\mu_0,\sigma_0} = \inf_{\tau \le 1} \mathsf{E}\left[\frac{2}{\sigma_0^3(1-\tau)} - |W_\tau + \mu_0/\sigma_0|\right]. \qquad \textbf{(B)}$$

(This "V_{μ_0,σ_0}-problem" was widely propagandized by L. Shepp and A. N. Shiryaev as an interesting **nonlinear** optimal stopping problem for Brownian motion, independently of Chernoff's problems.)

In the sequel we assume without loss of generality that $c = k = 1$.

THEOREM

1) Let τ_B^* be an optimal time in problem **(B)**.
Then optimal decision rule (τ_A^*, d_A^*) in problem **(A)** has the form

$$\tau_A^* = \frac{\tau_B^*}{\sigma_0^2(1-\tau_B^*)}, \qquad d_A^* = \operatorname{sgn}(X_{\tau_B^*} + \mu_0/\sigma_0^2).$$

2) Optimal time τ_B^* in problem **(B)** has the form

$$\tau_B^* = \inf\{0 \le t \le 1 : |W_t + \mu_0/\sigma_0| \ge a_{\sigma_0}(t)\},$$

where $a_{\sigma_0}(t)$ is a nonincreasing function on $[0,1]$ such that $a_{\sigma_0}(t) > 0$ for $t < 1$ and $a_{\sigma_0}(1) = 0$.

THEOREM (continued)

3) Function $a_{\sigma_0}(t)$ is a unique continuous solution of the integral equation

$$\frac{G(1-t,a(t))}{1-t} = \int_t^1 \frac{2}{\sigma_0^3(1-s)^2} \times \\ \times \left[\Phi\left(\frac{a(s)-a(t)}{\sqrt{s-t}}\right) - \Phi\left(\frac{-a(s)-a(t)}{\sqrt{s-t}}\right) \right] ds$$

in the class of functions $a(t)$ such that $a(t) \geq 0$ for $t < 1$ and $a(1) = 0$.

Here function $G(t,x)$ is defined in the following way:

$$G(t,x) = \frac{1}{\sqrt{t}}\varphi\Big(\frac{x}{\sqrt{t}}\Big) - \frac{|x|}{t}\Phi\Big(\frac{-|x|}{\sqrt{t}}\Big), \qquad t > 0, \quad x \in \mathbb{R},$$

where $\varphi(x)$, $\Phi(x)$is are standard normal density and distribution function.

REMARK

Chernoff has considered the process $X'_t = X_{t-1/\sigma_0^2} + \mu_0/\sigma_0^2$, which satisfies the equation

$$dX'_t = \frac{X'_t}{t}\,dt + dB'_t, \qquad t \geq 1/\sigma_0^2,$$

with some Brownian motion B'.

Then the optimal decision rule in problem (A) is obtained by finding the optimal time τ_C^* in the problem

$$V'(t,x) = \inf_{\tau \geq t} \mathsf{E}_{t,x}[\tau - G(\tau, X'_\tau)] \qquad \textbf{(C)}$$

for $t = 1/\sigma_0^2$, $x = \mu_0/\sigma_0^2$.

Optimal times τ_A^* and τ_C^* are connected by $\tau_A^* = \tau_C^* - 1/\sigma_0^2$. Optimal d_A^* equals $\operatorname{sgn}(X'_{\tau_C^*})$.

REMARK (continued)

Optimal time $\tau_C^* = \tau_C^*(x, t)$ in problem (C) is

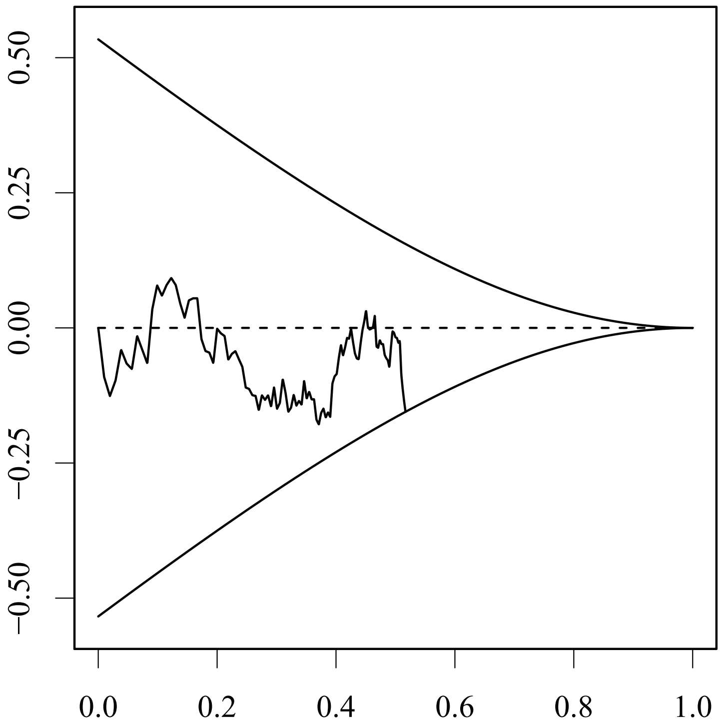
$$\tau_C^* = \inf\{s \geq t : |X'_s| \geq \gamma(s)\},$$

where $\gamma(s)$ is a certain strictly positive function for $t > 0$ (which does not depend on parameters μ_0, σ_0.)

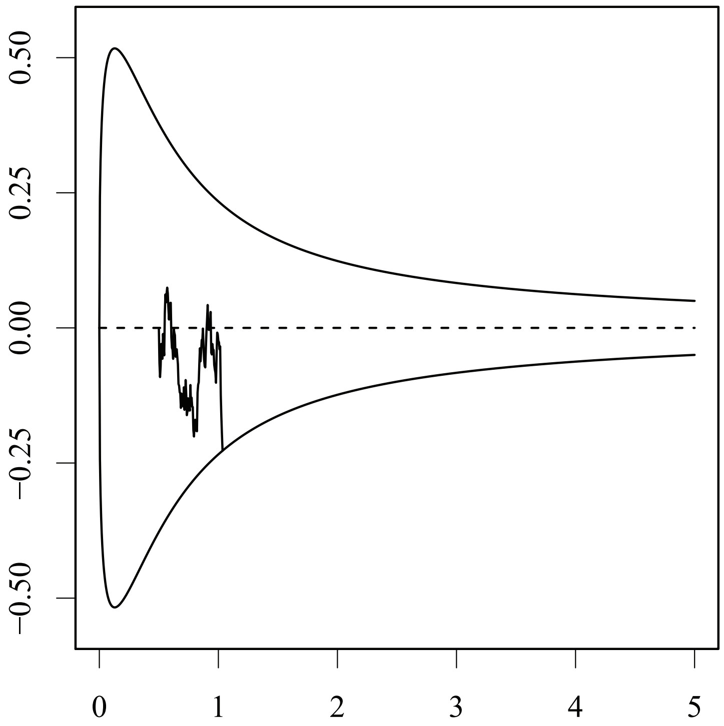
From the construction of processes W and X' we find that

$$\gamma(t) = \sigma_0 t \cdot a_{\sigma_0}(1 - 1/(\sigma_0^2 t)), \qquad t \geq 1/\sigma_0^2.$$

NUMERICAL SOLUTION



Boundary $a_{\sigma_0}(t)$ for $\sigma_0 = \sqrt{2}$
in Problem (B)



Boundary $\gamma(t)$
in Problem (C)

PROOF of the THEOREM

Step 1 (reduction to problem for Wiener process).

It suffices to consider decision rules (τ, d) with $\mathsf{E}\tau < \infty$. For any such rule we have

$$\mathcal{R}(\tau, d) = \mathsf{E}[\tau + \mathsf{E}(\mu^- \mid \mathscr{F}_\tau)\mathbb{I}\{d = +1\} + \mathsf{E}(\mu^+ \mid \mathscr{F}_\tau)\mathbb{I}\{d = -1\}].$$

Thus, we need to find time τ^* which minimizes the value

$$\mathscr{E}(\tau) = \mathsf{E}[\tau + \min\{\mathsf{E}(\mu^- \mid \mathscr{F}_\tau),\ \mathsf{E}(\mu^+ \mid \mathscr{F}_\tau)\}],$$

and to put

$$d^* = \begin{cases} +1, & \mathsf{E}(\mu^- \mid \mathscr{F}_{\tau^*}) \le \mathsf{E}(\mu^+ \mid \mathscr{F}_{\tau^*}), \\ -1, & \mathsf{E}(\mu^- \mid \mathscr{F}_{\tau^*}) > \mathsf{E}(\mu^+ \mid \mathscr{F}_{\tau^*}). \end{cases}$$

By the normal correlation theorem,

$$\mathscr{E}(\tau) = \mathsf{E}[\tau + G(\tau + 1/\sigma_0^2,\ X_\tau + \mu_0/\sigma_0^2)]$$

where $G(t,x)$ is the function

$$G(t,x) = \frac{1}{\sqrt{t}}\ \varphi(x/\sqrt{t}) - \frac{|x|}{t}\,\Phi(-|x|/\sqrt{t}),$$

already introduced above.

The innovation representation for X implies

$$dX_t = \mathsf{E}(\mu\,|\,\mathcal{F}_t)\,dt + d\bar{B}_t \quad \Rightarrow \quad dX_t = \frac{X_t + \mu_0/\sigma_0^2}{t + 1/\sigma_0^2}\,dt + d\bar{B}_t.$$

with Brownian motion $\bar{B}_t = X_t - \int_0^t \mathsf{E}(\mu\,|\,\mathcal{F}_s)\,ds$.

In particular, X is a **Markov process**.

Direct calculations yield

$$\boxed{\mathscr{L}_{t,x}[G(t,x) + |x|/2t] = 0}$$

where

$$\mathscr{L}_{t,x} = \frac{\partial}{\partial t} + \frac{x + \mu_0/\sigma_0^2}{t + 1/\sigma_0^2} \cdot \frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}.$$

Then for any stopping time τ, $\mathsf{E}\tau < \infty$, by applying the Itô formula to the expression

$$\mathscr{E}(\tau) = \mathsf{E}[\tau + G(\tau + 1/\sigma_0^2, X_\tau + m_0/\sigma_0^2)],$$

we find

$$\mathscr{E}(\tau) = \mathsf{E}\left[\tau - \frac{|X_\tau + m_0/\sigma_0^2|}{2(\tau + 1/\sigma_0^2)}\right] + G\left(\frac{1}{\sigma_0^2}, \frac{m_0}{\sigma_0^2}\right) + \frac{|m_0|}{2}.$$

Also by direct calculation we get that

the process $M_t = \dfrac{X_t + m_0/\sigma_0^2}{\sigma_0(t + 1/\sigma_0^2)} - \dfrac{m_0}{\sigma_0}$ **is a martingale**.

Using a change of time, we find that

the process $W_t = M_{t/\sigma_0^2(1-t)}$ **is a Brownian motion**.

Then for any stopping time τ such that $\mathsf{E}\tau < \infty$ we have

$$\mathscr{E}(\tau) = \frac{\sigma_0}{2}\mathsf{E}\left[\frac{2}{\sigma_0^3(1-\tau_B)} - |W_{\tau_B} + \mu_0/\sigma_0|\right] + \;\ldots\ldots$$

where ...... is the deterministic part which does not depend on τ, τ_B is a stopping time associated with τ by the formula

$$\tau_B = \frac{\sigma_0^2\tau}{1+\sigma_0^2\tau}.$$

Thus, to find optimal decision rule (τ_A^*, d_A^*) in the initial problem of distinguishing between H^+ and H^- it suffices to find optimal time τ_B^* in problem

$$V_{\mu_0,\sigma_0} = \inf_{\tau \le 1} \mathsf{E}\left[\frac{2}{\sigma_0^3(1-\tau)} - |W_\tau + \mu_0/\sigma_0|\right] \qquad \textbf{(B)}$$

and to put

$$\tau_A^* = \frac{\tau_B^*}{\sigma_0^2(1-\tau_B^*)}, \qquad d_A^* = \operatorname{sgn}(X_{\tau_B^*} + \mu_0/\sigma_0^2).$$

Step 2 (analysis of the structure of the optimal time in problem (B)).

For the solution of problem (B) consider the **value function**

$$V(t,x) = \inf_{\tau \le 1-t} \mathsf{E}\left[\frac{2/\sigma_0^2}{1-(\tau+t)} - |W_\tau + x|\right] - \frac{2/\sigma_0^2}{1-t},$$

letting $V(1,x) = 0$ for all x.

One can prove that $\boldsymbol{V(t,x)}$ **is continuous**, and optimal stopping time has the form

$$\tau^*(t,x) = \inf\{s \ge 0 : (s+t, W_s + x) \notin C\},$$

where C is the **set of continuation of observation**:

$$C = \{(t,x) : V(t,x) < -|x|\}$$

($-|x|$ is a gain from instantaneous stopping).

Analyzing the structure of $V(t, x)$, we establish that

$$\boxed{C = \{(t, x) : t \in [0, 1),\ |x| < a(t)\}},$$

where $a(t)$ is some nonincreasing function on $[0, 1]$ such that $a(t) > 0$ for $t < 1$ and $a(1) = 0$.

Moreover, one can prove that **$a(t)$ is continuous** on $[0, 1]$.

Step 3 (integral equation).

Using the general theory of optimal stopping, one can prove that $V(t,x)$ solves the following problem for the operator $\mathscr{L}_{t,x}$:

$$\begin{cases} \mathscr{L}_{t,x}V(t,x) = -\dfrac{2/\sigma_0^3}{(1-s)^2}, & |x| < a(t), \\ \dfrac{\partial V}{\partial x}(t,x) = -\operatorname{sgn}(x), & x = \pm a(t), \\ V(t,x) = -|x|, & |x| \geq a(t). \end{cases}$$

Applying the Itô formula gives

$$\mathsf{E}V(1, W_{1-t}+x) = V(t,x) \\ + \int_t^1 \mathscr{L}_{t,x}V(s, W_{1-s}+x) \cdot \mathbb{I}(|W_{1-s}+x| \neq a(s))\,du.$$

Using equalities $V(1,x) = -|x|$ for all $x \in \mathbb{R}$,
$\mathscr{L}_{t,x} V(t,x) = 0$ for $|x| > a(t)$, we get

$$V(t,x) = -\mathsf{E}|W_{1-t} + x| + \int_t^1 \frac{2/\sigma_0^2}{(1-s)^2} \mathsf{P}(|W_{1-s} + x| < a(s))\, ds.$$

Using equality $V(t, a(t)) = -a(t)$, we find

$$\mathsf{E}|W_{1-t} + a(t)| - a(t) = \int_t^1 \frac{2/\sigma_0^2}{(1-s)^2} \mathsf{P}(|W_{1-s} + a(t)| < a(s))\, ds,$$

which, after calculation of E$|\ldots|$ and P$(\ldots)$, turns into the required equation.

Step 4 (uniqueness of solution of the integral equation).

Proof follows the method of:

P.V.Gapeev, G.Peskir. The Wiener disorder problem with finite horizon (Stochastic Process. Appl. 116:2 (2006))

G.Peskir, A.N.Shiryaev. Optimal stopping and free-boundary problems (Birkhäuser, 2006)

§ 4. Distinguishing between three hypotheses

We observe a random process

$$X_t = \mu t + B_t,$$

where μ is a random variable, which does not depend on B and takes values m_0, m_1, m_2 with probabilities π^0, π^1, π^2.

Bayesian risk:

$$\boxed{\mathcal{R}(\tau, d) = \mathsf{E}[c\tau + W(\mu, d)]}$$

where $c > 0$ is a constant, $W(\mu, d)$ is a **penalty function**:

$$W(m_i, m_i) = 0, \qquad i = 0, 1, 2,$$
$$W(m_i, m_j) = a_{ij}, \qquad i, j = 0, 1, 2, \quad i \neq j,$$

with $a_{ij} > 0$.

For simplicity, let $m_1 = -1$, $m_0 = 0$, $m_2 = 1$, $a_{ij} = 1$, $\pi^i = 1/3$.

Introduce the **process of a posteriori probabilities** $\pi^i = (\pi^i_t)_{t \geq 0}$:

$$\pi^i_t = \mathsf{P}(\mu = m_i \,|\, \mathcal{F}^X_t), \qquad i = 0, 1, 2.$$

Then for any decision rule (τ, d), $\mathcal{R}(\tau, d)$ takes the form

$$\mathcal{R}(\tau, d) = \mathsf{E}_\pi \Big[c\tau + 1 - \sum_i \pi^i_\tau \mathbb{I}\{d = \mu_i\} \Big]$$

Consequently, we must **find a time τ^* which minimizes**

$$\boxed{\mathsf{E}_\pi[c\tau + 1 - \max\{\pi^0_\tau,\ \pi^1_\tau,\ \pi^2_\tau\}]}$$

and **define d^* by the formula**

$$\boxed{d^* = m_i, \quad \text{where} \quad i = \operatorname{argmax}_i \pi^i_\tau}$$

Our problem reduces to the **problem of optimal stopping of the observed process X**.

From the **innovation representation** for X we obtain

$$dX_t = \mathsf{E}(\mu \mid \mathcal{F}_t^X)\, dt + d\bar{B}_t,$$

where $\bar{B}_t = X_t - \int_0^t \mathsf{E}(\mu \mid \mathcal{F}_s^X)\, ds$ is a Brownian motion.

The properties of conditional expectation yield

$$\mathsf{E}(\mu \mid \mathcal{F}_t^X) = \mu_0\pi_t^0 + \mu_1\pi_t^1 + \mu_2\pi_t^2 = \pi_t^2 - \pi_t^1.$$

Calculating π_t^i by means of the **Bayes formula** gives

$$dX_t = \frac{e^{-t/2}(e^{X_t} - e^{-X_t})}{1 + e^{-t/2}(e^{X_t} + e^{-X_t})}\, dt + d\bar{B}_t.$$

Thus, the problem

$$\boxed{\inf_\tau \mathsf{E}[c\tau + G(\pi^0_\tau, \pi^1_\tau, \pi^2_\tau)]}$$

with

$$G(\pi^0_\tau, \pi^1_\tau, \pi^2_\tau) = \min\{\pi^1_\tau + \pi^2_\tau,\ \pi^0_\tau + \pi^2_\tau,\ \pi^0_\tau + \pi^1_\tau\}$$

is replaced by the problem

$$\boxed{\inf_\tau \mathsf{E}[c\tau + G(\tau, X_\tau)]}$$

with

$$G(t,x) = \frac{\min(e^x + e^{-x}, 1 + e^x, 1 + e^{-x})}{1 + e^{-t/2}(e^x + e^{-x})}.$$

Following the general theory, introduce the **value function** in problem

$$V(t,x) = \inf_{\tau} \mathsf{E}_{t,x}[c\tau + G(\tau + t, X_{t+\tau})]$$

Optimal stopping time is

$$\tau^*(t,x) = \inf_{\tau}\{s \geq 0 : V(t+s, X_{t+s}) = G(t+s, X_{t+s})\}.$$

Now we characterize the set of **continuation of observation**

$$C = \{(t,x) : V(t,x) < G(t,x)\}$$

for “large” t.

THEOREM 1 *(qualitative behavior of stopping boundaries)*

There exist $T_0 > 0$ and functions $f(t)$, $g(t)$ such that the set

$$C_{\geq T_0} = \{(t, x) \in C : t \geq T_0\}$$

admits the representation

$$C_{\geq T_0} = \Big\{(t, x) : t \geq T_0 \text{ and } |x| \in (g(t), f(t))\Big\}.$$

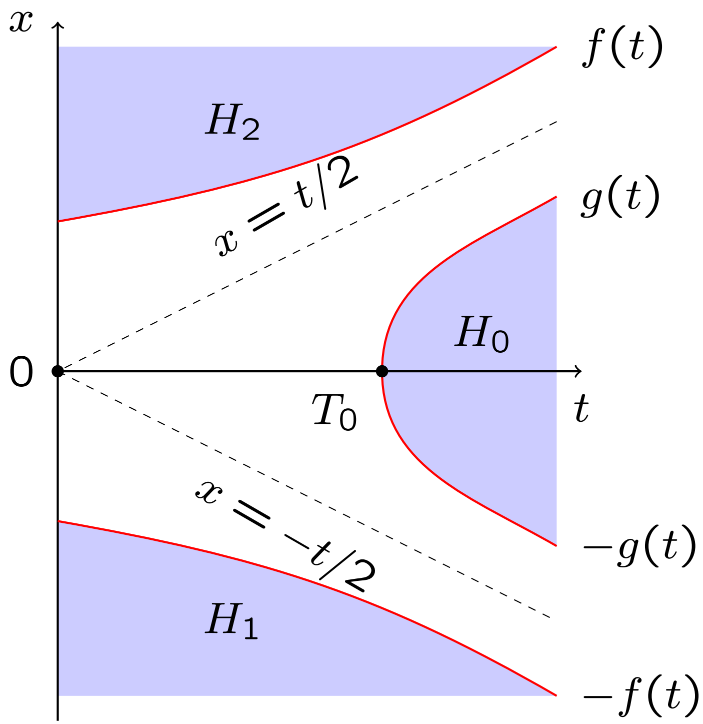
Functions $f(t)$ and $g(t)$ are such that

$$f(t) = t/2 + b + O(e^{-t}), \quad g(t) = t/2 - b + O(e^{-t}),$$

where the constant b is a unique solution of the equation

$$e^b - e^{-b} + 2b = 1/(2c).$$

OPTIMAL STOPPING BOUNDARIES



The set of continuation of observation has the property

$$C_{\geq T_0} = \{(t, x) : t \geq T_0 \text{ and } |x| \in (g(t), f(t))\}.$$

THEOREM 2 *(integral equations)*

For all $t \geq T_0$ stopping boundaries $f(t)$, $g(t)$ satisfy the system of integral equations

$$\begin{cases} c\int_t^\infty K_1(f(t),t,s,f(s),g(s))ds = \int_t^\infty K_2(f(t),t,s)ds \\ c\int_t^\infty K_1(g(t),t,s,f(s),g(s))ds = \int_t^\infty K_2(g(t),t,s)ds \end{cases}$$

where function K_1 and K_2 are defined by

$$K_1(x,t,s,f,g) = \frac{\sum_i [\Phi_{s-t}(f-x-\mu_i(s-t)) - \Phi_{s-t}(g-x-\mu_i(s-t))]\,\varphi_t(x-\mu_i t)}{\sum_j \varphi_t(x-\mu_j t)}$$

$$K_2(x,t,s) = \frac{\sum_i \varphi_{s-t}(\mu_i(s-t)-s/2+x)\,\varphi_t(x-\mu_i t)}{2(2+e^{-s})\sum_j \varphi_t(x-\mu_j t)},$$

where $\varphi_r(y) = \frac{1}{\sqrt{2\pi r}}e^{-y^2/(2r)}$ and $\Phi_r(z) = \int_{-\infty}^z \varphi_r(y)dy$.

§ 5. Disorder problem on finite intervals

We observe a process $X = (X_t)_{t \geq 0}$,

$$\boxed{X_t = \mu(t - \theta)^+ + B_t},$$

where θ is a random variable which does not depend on B and is **UNIFORMLY** distributed on $[0, 1]$.

We consider the following problems:

$$V_1 = \inf_{\tau \leq 1} \Big[\mathsf{P}(\tau < \theta) + c\mathsf{E}(\tau - \theta)^+ \Big],$$

$$V_2 = \inf_{\tau \leq 1} \mathsf{E}|\tau - \theta|.$$

The key point to solution of problems V_1 and V_2 is **reduction to Markovian problems of optimal stopping**.

Introduce the **Shiryaev–Roberts statistic** $\psi = (\psi_t)_{t\geq 0}$:

$$\psi_t = e^{\mu X_t - \mu^2 t/2} \int_0^t e^{-\mu X_s + \mu^2 s/2}\, ds,$$

or, in differentials,

$$d\psi_t = dt + \mu\psi_t\, dX_t, \qquad \psi_0 = 0.$$

Process ψ_t is related to process of **a posteriori probabilities** $\pi_t = \mathsf{P}(\theta \leq t \,|\, \mathcal{F}_t^X)$ by the following formula:

$$\psi_t = \frac{\pi_t}{1-\pi_t}(1-t).$$

Lemma

The following representations hold:

$$V_1 = \inf_{\tau \le 1} \mathsf{E}^{\infty} \left[\int_0^{\tau} (c\psi_s - 1)\, ds \right] + 1,$$

$$V_2 = \inf_{\tau \le 1} \mathsf{E}^{\infty} \left[\int_0^{\tau} (\psi_s - (1 - s))\, ds \right],$$

where $\mathsf{E}^{\infty}[\,\cdot\,]$ stands for the expectation in absence of disorder (i. e., when X is a Brownian motion).

Proof is based on the following equalities:

$$\mathsf{E}(\tau - \theta)^+ = \mathsf{E}^{\infty} \left[\textstyle\int_0^{\tau} \psi_s \, ds \right],$$

$$\mathsf{P}(\tau < \theta) = 1 - \mathsf{E}^{\infty}\tau,$$

$$\mathsf{E}(\tau - \theta)^- = \mathsf{E}^{\infty}(1 - \tau)^2/2.$$

Proof of the lemma

1) Rewrite the average time of delay $\mathsf{E}(\tau-\theta)^+$:

$$
\begin{aligned}
\mathsf{E}(\tau-\theta)^+ &= \int_0^1 \mathsf{E}[(\tau-u)^+ \,|\, \theta=u]\,du \\
&= \int_0^1 \int_u^1 \mathsf{E}[\mathbb{I}(s\le\tau)|\theta=u]\,ds \\
&= \int_0^1 \int_u^1 \mathsf{E}^\infty[\mathbb{I}(s\le\tau)e^{\mu(X_s-X_u)-\mu^2(s-u)/2}]\,ds \\
&= \mathsf{E}^\infty \int_0^\tau \int_0^s e^{\mu(X_s-X_u)-\mu^2(s-u)/2}\,ds \\
&= \int_0^\tau \psi_s\,ds.
\end{aligned}
$$

2) Rewrite the probability of a false alarm $\mathsf{P}(\tau < \theta)$:

$$\begin{aligned}\mathsf{P}(\tau < \theta) &= \int_0^1 \mathsf{P}(\tau < u \,|\, \theta = u)\, du \\ &= \int_0^1 \mathsf{P}^\infty(\tau < u)\, du \\ &= \mathsf{E}^\infty \tau\end{aligned}$$

3) Rewrite the average time after a false alarm $\mathsf{E}(\tau - \theta)^-$:

$$\begin{aligned}\mathsf{E}(\tau - \theta)^- &= \int_0^1 \mathsf{E}[(\tau - u)^- \,|\, \theta = u]\, du \\ &= \int_0^1 \mathsf{E}^\infty(\tau - u)^-\, du \\ &= \mathsf{E}^\infty(1-\tau)^2/2\end{aligned}$$

$\square$

Thus, for the initial problems

$$V_1 = \inf_{\tau \le 1} \left[\mathsf{P}(\tau < \theta) + c\mathsf{E}(\tau - \theta)^+ \right], \qquad V_2 = \inf_{\tau \le 1} \mathsf{E}|\tau - \theta|$$

we got the representations

$$V_1 = \inf_{\tau \le 1} \mathsf{E}^\infty \left[\int_0^\tau (c\psi_s - 1)\, ds \right] + 1,$$
$$V_2 = \inf_{\tau \le 1} \mathsf{E}^\infty \left[\int_0^\tau (\psi_s - (1 - s))\, ds \right],$$

where ψ has the differential

$$d\psi_t = dt + \mu\psi_t\, dX_t, \qquad \psi_0 = 0,$$

and X_t is a Brownian motion w.r.t. P^∞.

Introduce functions $f_1(t) = 1/c$ and $f_2(t) = 1 - t$.

Theorem

Optimal stopping times for V_1 and V_2 are

$$\boxed{\tau_i^* = \inf\{t \geq 0 : \psi_t \geq a_i^*(t)\} \wedge 1, \qquad i = 1, 2}$$

where $a_i^*(t)$ is a unique continuous solution of the equation

$$\int_t^1 \mathsf{E}^{\infty}\Big[(\psi_s - f_i(s))\mathbb{I}\{\psi_s \leq a_i^*(s)\} \,\Big|\, \psi_t = a_i^*(t)\Big]\, ds = 0,$$

satisfying the conditions

$$a_i^*(t) \geq f_i(t) \text{ for } t < 1, \qquad a_i^*(1) = f_i(1).$$

Theorem (continued)

Values V_1 and V_2 are given by

$$V_1 = \int_0^1 \mathsf{E}^\infty (c\psi_s - 1)\mathbb{I}\{\psi_s < a_1^*(s)\}\, ds + 1,$$
$$V_2 = \int_0^1 \mathsf{E}^\infty [\psi_s - (1 - s)]\mathbb{I}\{\psi_s < a_2^*(s)\}\, ds.$$

Proof of the theorem

For the solution of the problem, consider the **value function**

$$V_i(t,x) = \inf_{\tau \le 1-t} \mathsf{E}_x^\infty \left[\int_0^\tau (\psi_s - f_i(t+s))\, ds \right], \qquad i = 1, 2.$$

where $\mathsf{E}_x^\infty[\,\cdot\,]$ stands for expectation under assumption $\psi_0 = x$.

One can prove that $V_i(t,x)$ are continuous, and optimal stopping times have the form

$$\tau_i^*(t,x) = \inf\{s \ge 0 : (t+s, \psi_s) \notin C_i\},$$

where C_i is the **set of continuation of observations**:

$$C = \{(t,x) : V_i(t,x) < 0\}$$

(here 0 is a gain from instantaneous stopping).

Analyzing the structure of functions $V_i(t,x)$, we establish that

$$C_i = \{(t,x) : t \in [0,1),\ x < a_i^*(t)\},$$

where $a_i^*(t)$ are unknown nonincreasing functions on $[0,1]$, at that $a_i(t) \geq f_i(t)$ for $t < 1$ and $a_i(1) = f_i(1)$.

One can prove that $a_i(t)$ are **continuous** on $[0,1]$.

One can prove also that $V_i(t,x)$ solves a **free-boundary problem**

$$\begin{cases} V_t'(t,x) + \mathscr{L}_\psi V(t,x) = f_i(t) - x, & x < a_i(t), \\ V(t,x) = 0, & x \geq a_i(t), \\ V(t,x-) = 0, & x = a_i(t), \\ V_{x-}'(t,x) = 0, & x = a_i(t), \end{cases}$$

where

$$\mathscr{L}_\psi = \frac{\mu^2 x^2}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}.$$

Applying the Itô formula to $V_i(s,\psi_s)$, we get

$$\mathsf{E}_x^\infty V(1,\psi_{1-t}) = V(t,x) + \mathsf{E}_x^\infty \int_0^{1-t} [V_t' + \mathscr{L}_\psi V](t+s,\psi_s)\cdot \mathbb{I}(\psi_s < a(t+s))\, ds$$

Since $V_i(1,\cdot)\equiv 0$, and $V_i(t,x)=0$ for $x=a_i^*(t)$, we find

$$V(t,x) = -\mathsf{E}_x^\infty \int_0^{1-t} [V_t' + \mathscr{L}_\psi V](t+s,\psi_s)\cdot \mathbb{I}(\psi_s < a(t+s))\, ds,$$

which gives, after substitution of $[V_t' + \mathscr{L}_\psi V](t,x) = f_i(t) - x$, the required equation.

Proof of **uniqueness of solution** of the integral equations is given in (Zhitlukhin, Shiryaev, TPA, 2012).

Numerical results

Integral equation

$$\int_t^1 \mathsf{E}^\infty\Big[(\psi_s - f_i(s))\mathbb{I}\{\psi_s \le a_i^*(s)\} \,\Big|\, \psi_t = a_i^*(t)\Big]\, ds = 0 \qquad (*)$$

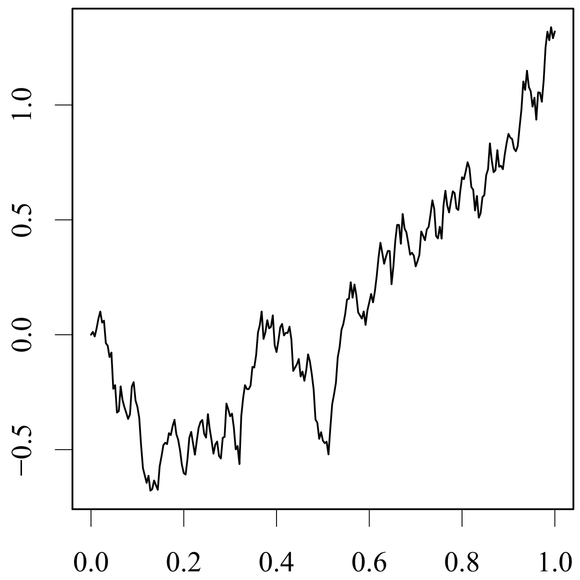
can be solved numerically by **“backward induction”**:

1. Fix the partition $0 = t_0 < t_1 < \ldots < t_n = 1$;
2. Take $a_i^*(t_n) = f_i(1)$ (by the theorem);
3. If $a_i^*(t_k), \ldots, a_i^*(t_n)$ are calculated, then we find $a_i^*(t_{k-1})$ by
   - calculating integral $\int_{t_{k-1}}^1$ in $(*)$ with stepwise function equal to $a_i^*(\cdot)$ in points $t_k, \ldots, t_n$ and
   - solving the resulting algebraic equation w.r.t. $a_i^*(t_{k-1})$.

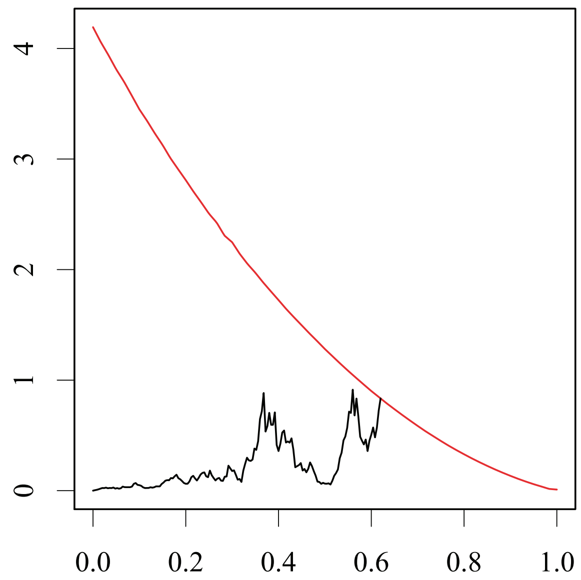
Example

For $\mu = 4$ consider the problem

$$V_2 = \inf_{t \le 1} \mathsf{E}|\tau - \theta|.$$



process X_t; $\theta = 0.5$.



process ψ_t and boundary $a_2^*(t)$.

§ 6. Disorder and Finance. I: Bubbles

We observe **Brownian motion with disorder** $(X_t)_{t\geq 0}$:

$$\boxed{dX_t = [\mu_1 \mathbb{I}(t < \theta) + \mu_2 \mathbb{I}(t \geq \theta)]\, dt + \sigma\, dB_t}$$

where $\theta \sim U[0,1]$, $\mu_1 > 0 > \mu_2$ (in case of long position), $\mu_1 < 0 < \mu_2$ (in case of short position), $\sigma > 0$ (drift changes from μ_1 to μ_2). We restrict our analysis to the case of long position only.

Below we consider **problems of optimal stopping**:

$$H_{\mathrm{I}} = \sup_{\tau \leq 1} \mathsf{E} X_\tau, \qquad H_{\mathrm{II}} = \sup_{\tau \leq 1} \mathsf{E} \exp(X_\tau - \sigma^2 \tau/2).$$

Earlier problems of such type were considered in *(Beibel, Lerche, 1997), (Shiryaev, Novikov, 2008), (Ekström, Lindberg, 2012)*.

Application in mathematical finance

Let the **price of an asset** be modeled by geometrical Brownian motion with disorder $S_t = \exp(X_t - \sigma^2 t/2)$:

$$dS_t = [\mu_1 \mathbb{I}(t < \theta) + \mu_2 \mathbb{I}(t \geq \theta)] S_t \, dt + \sigma S_t \, dB_t, \qquad S_0 = 1,$$

i. e., the price in average grows up “till” time θ, and falls down “after” θ.

Problem H_{I} consists in **maximization of logarithmic utility** of selling asset:

$$H_{\mathrm{I}} = \sup_{\tau \leq 1} \mathsf{E}(\log S_\tau), \qquad [\text{для } \mu_i' = \mu_i - \sigma^2/2].$$

Problem H_{II} consists in **maximization of linear utility** of selling asset:

$$H_{\mathrm{II}} = \sup_{\tau \leq 1} \mathsf{E} S_\tau.$$

Solution of the problem H_l

Since $X_t = \mu_1 t + (\mu_2 - \mu_1)(t-\theta)^+ + \sigma B_t$, we have for any stopping time $\tau \le 1$

$$\mathsf{E} X_\tau = \mathsf{E}[\mu_1 \tau - (\mu_1 - \mu_2)(\tau - \theta)^+].$$

Denoting $\mu = (\mu_1 - \mu_2)/\sigma$ and $\widetilde{X} = (X_t - \mu_1 t)/\sigma$, we find

$$\psi_t = e^{-\mu \widetilde{X}_t - \mu^2 t/2} \int_0^t e^{\mu \widetilde{X}_s + \mu^2 s/2}\, ds.$$

Analogously to the result above,

$$H_{\mathrm{I}} = \sup_{\tau \le 1} \mathsf{E}^\infty \left[\int_0^\tau (\mu_1 - (\mu_1 - \mu_2)\psi_s)\, ds \right],$$

where $\mathsf{E}^\infty[\,\cdot\,]$ stands for expectation under assumption that $\widetilde{X}$ is a standard Brownian motion.

Theorem

Optimal stopping time in problem $H_{\rm I}$ is

$$\boxed{\tau_l^* = \inf\{t \ge 0 : \psi_t \ge a_l^*(t)\} \wedge 1}$$

where $a_l^*(t)$ is a unique continuous solution of the equation

$$\int_t^1 \mathsf{E}^\infty\Big[(\mu_1 - (\mu_1 - \mu_2)\psi_s)\mathbb{I}(\psi_s \le a_l^*(s)) \,\Big|\, \psi_t = a_l^*(t)\Big]\, ds = 0,$$

satisfying the conditions

$$a_l^*(t) \ge \frac{\mu_1}{\mu_1 - \mu_2} \text{ for } t < 1, \qquad a_l^*(1) = \frac{\mu_1}{\mu_1 - \mu_2}.$$

The value $H_{\rm I} = \mathsf{E}X_{\tau_l^*}$ can be found by the formula

$$H_{\rm I} = \int_0^1 \mathsf{E}^\infty[\mu_1 - (\mu_1 - \mu_2)\psi_s]\mathbb{I}(\psi_s < a_l^*(s))\, ds.$$

Solution of problem H_g

We introduce a **new measure** $\widetilde{\mathsf{P}}$ such that

$$(\widetilde{X}_t - \sigma t) \text{ is a } \widetilde{\mathsf{P}}\text{-Brownian motion,}$$

where $\widetilde{X} = (X_t - \mu_1 t)/\sigma$.

We establish that for any stopping time $\tau \le 1$

$$\mathsf{E}^{\mathsf{P}} S_\tau = \mathsf{E}^{\widetilde{\mathsf{P}}} \left[S_\tau \times \frac{d\mathsf{P}_\tau}{d\widetilde{\mathsf{P}}_\tau} \right] = \mathsf{E}^{\widetilde{\mathsf{P}}} \Big[e^{\mu_1 \tau} (\psi_\tau + 1 - \tau) \Big],$$

at that process ψ has differential

$$d\psi_t = [1 - (\mu_1 - \mu_2)\psi_t]\, dt + \mu \psi_t \, d(\widetilde{X}_t - \sigma t), \qquad \psi_0 = 0.$$

Applying the Itô formula, we get

$$\mathsf{E}^{\mathsf{P}} S_\tau = \mathsf{E}^{\widetilde{\mathsf{P}}} \left[\int_0^\tau e^{\mu_1 s} (\mu_2 \psi_s + \mu_1 (1 - s)) \, ds \right] + 1.$$

Theorem

Optimal stopping time in problem H_{II} is

$$\boxed{\tau_g^* = \inf\{t \geq 0 : \psi_t \geq a_g^*(t)\}}$$

where $a_g^*(t)$ is a unique continuous solution of the equation

$$\int_t^1 \mathsf{E}^{\widetilde{\mathsf{P}}}\Big[(\mu_2\psi_s + \mu_1(1-s))\mathbb{I}(\psi_s \leq a_g^*(s)) \,\Big|\, \psi_t = a_g^*(t)\Big]\, ds = 0,$$

satisfying the conditions

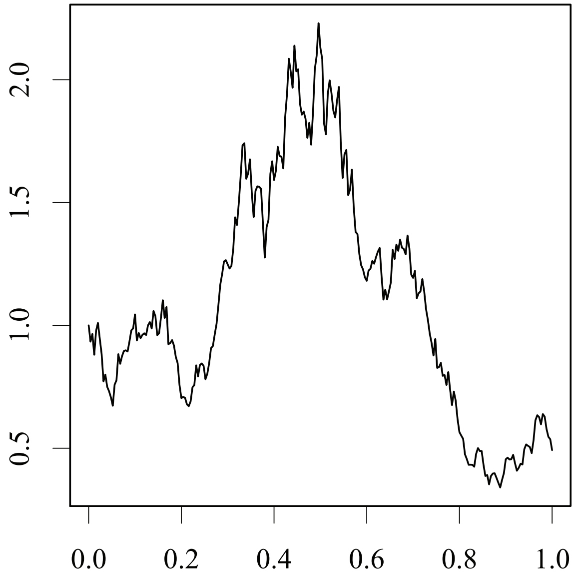
$$a_g^*(t) \geq \frac{\mu_1}{|\mu_2|}(1-t) \text{ for } t < 1, \qquad a_g^*(1) = 0.$$

The value $H_{\mathrm{II}} = \mathsf{E} S_{\tau_g^*}$ van be found by the formula

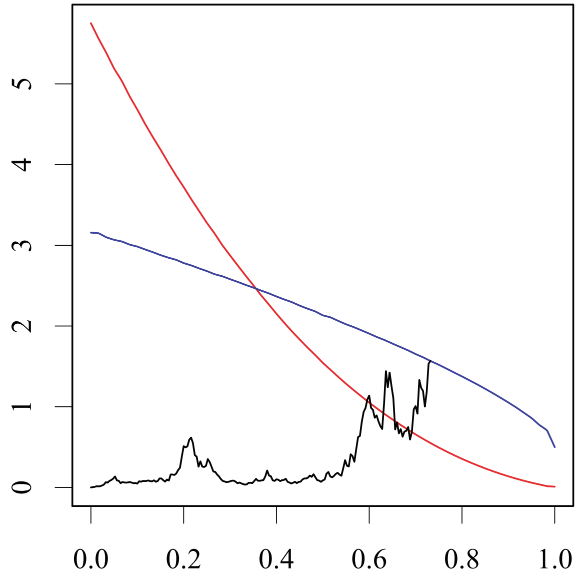
$$H_{\mathrm{II}} = \int_0^1 \mathsf{E}^{\widetilde{\mathsf{P}}}[\mu_2\psi_s + \mu_1(1-s)]\mathbb{I}(\psi_s < a_g^*(s))\, ds + 1.$$

Example

Consider problems H_{I} and H_{II} for $\mu_1 = -\mu_2 = 2$, $\sigma = 1$.



process S_t; $\theta = 0.5$.



ψ_t and boundaries $a_l^*(t)$, $a_h^*(t)$.

III. When to sell Apple?

Let us apply our results to problems of mathematical finance **based on real asset prices**.

Consider two “bubbles” on financial markets:

- Increase of prices of Apple assets from 2009 to 2012.

- Increase of prices of Internet companies assets at the end of 1990’s.

Problem consists in choosing optimal time of exit from “bubble” with maximum gain.

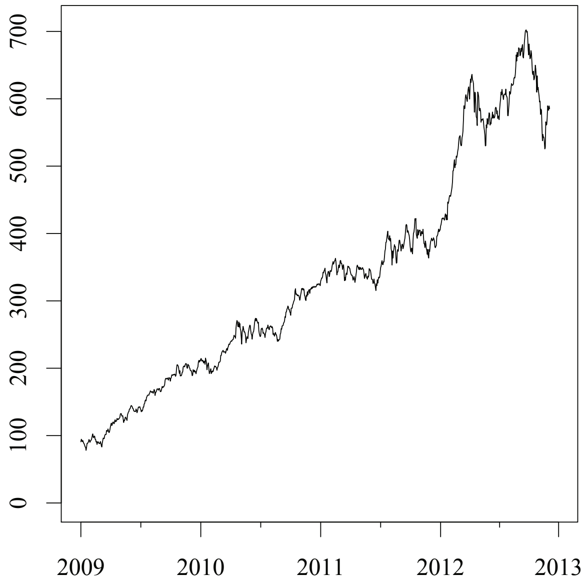
REMARK. The basic idea of bubbles is that there is a **FAST** rate of growth in prices, then **PEAK**, and then a fast **DECLINE**. There are several papers of Robert Jarrow and Philip Protter (see, e.g., SIAM J. Financial Math., 2 (2011), 839–865), where they developed the "martingale theory of bubbles". Their analysis is based on idea that **prices of bubbles behave similarly to the path behavior of the "strict nonnegative continuous local martingale"**. A typical path of such processes is to shoot up to high value and then quickly decrease to small values and remain at them. Jarrow and Protter proposed some "stochastic volatility models", saying that appearing of bubbles in prices relates with increasing of the volatility.

Our analysis of bubbles is based on idea of work with **drift terms (increasing/decreasing)**.

Example 1. Increase of Apple asset prices

In 2009–2012 prices on Apple assets grew up in almost 9 times. Minimum equals \$82.33 (6/03/09), maximum equals \$705.07 (21/09/12).

However, already on 15/11/12 the price fell down to \$522.62.



The fall down at the end of 2012 was expected already at the beginning of the year.

Setting of the problem of optimal exit from "bubble"

Agents on the market might not be aware of existence of a probability-statistical model of price evolution.

From their point of view, the question considered sounds as follows:

1. One observe a sequence of prices

$$P_0, P_1, \ldots, P_N,$$

where P_0 is price on 6/03/09 and P_N is price on 31/12/12.

2. One expect prices to fall down at the end of 2012

3. For a given date $n_0 < N$ of buying asset, one wants to find a time of selling it which would maximize the gain.

Representation of observed prices by process with disorder

1. We project dates $n_0, \ldots, N$ onto the interval $[0,1]$, since one market day has length $\Delta t = 1/(N - n_0)$.

Assume that prices are modeled by process

$$dS_t = [\mu_1 \mathbb{I}(t < \theta) + \mu_2 \mathbb{I}(\theta \geq t)] S_t \, dt + \sigma S_t \, dB_t,$$

where $S_{k\Delta t} = P_k / P_0$ and $\theta \sim U[0,1]$.

2. Parameters μ_1 and σ are estimated from data $P_0, \ldots, P_{n_0}$.

The choice of μ_2 is subjective but $\mu_2 = -\mu_1$ is proved empirically to be good (one can see it from other cases).

3. Then one applies results on solution of the problem of maximization of $\mathsf{E} S_\tau$.

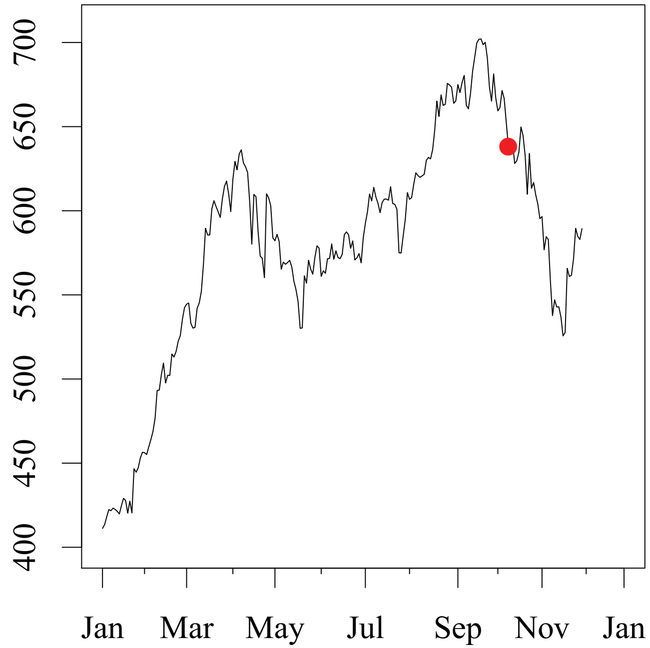
Results of choice of time for selling Apple

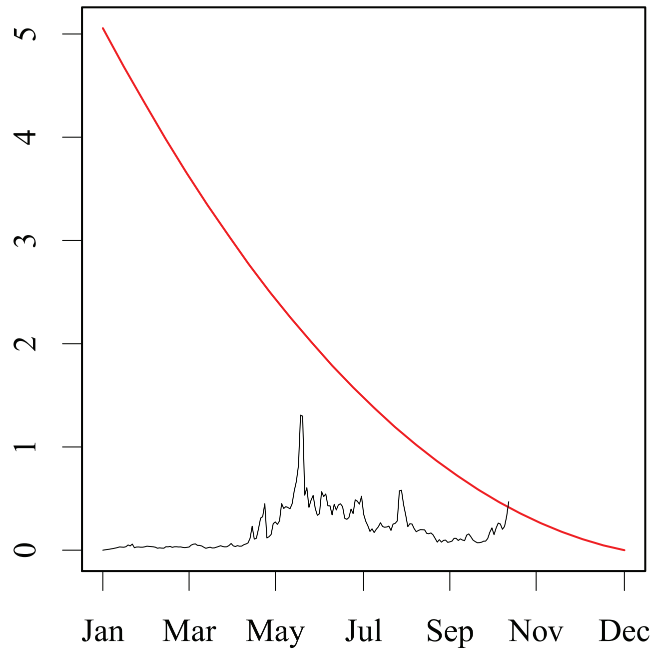
Buy	Sell
3-Jan-11 ($ 329.57)	9-Oct-12 ($ 635.85)
1-Jul-11 ($ 343.26)	8-Oct-12 ($ 638.17)
3-Jan-12 ($ 411.23)	8-Oct-12 ($ 638.17)
1-May-12 ($ 582.13)	9-Oct-12 ($ 635.85)
3-Jul-12 ($ 599.41)	9-Oct-12 ($ 635.85)
1-Aug-12 ($ 606.81)	11-Oct-12 ($ 628.10)

Results of the work of our method in case when assets were bought on 3 January 2012.

On the left are prices (red point = time of selling).

On the right are statistic ψ and optimal stopping boundary.

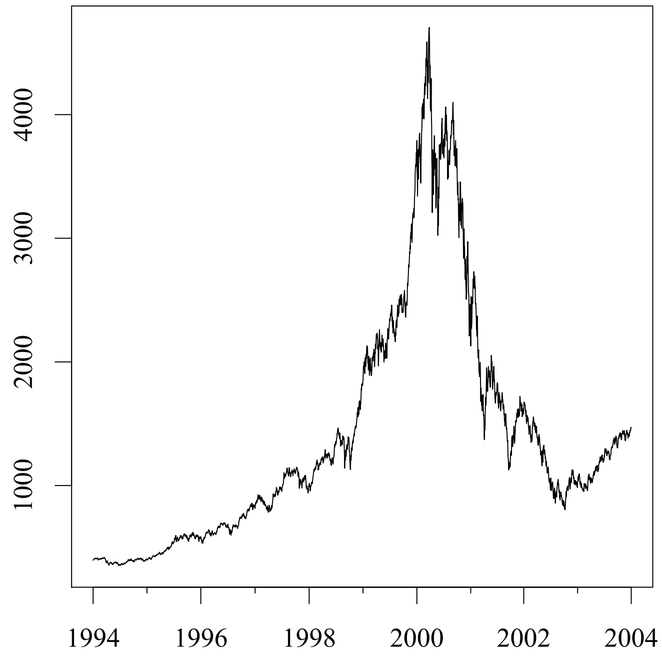




Example 2. Rise of NASDAQ index

• From the beginning of 1994 till March 2000, NASDAQ-100 grew up in more than 12 times, from 395.53 to 4816.35. Then it fell down in 6 times, to 795.25, by October 2002

• For example, the Soros Foundation has lost $ 5 bln. of $ 12 bln.



Results of choice of time for selling NASDAQ-100

Buy	**Sell**
2-Jul-98 ($ 1332.53)	12-Apr-00 ($ 3633.63)
4-Jan-99 ($ 1854.39)	13-Apr-00 ($ 3553.81)
1-Jul-99 ($ 2322.32)	13-Apr-00 ($ 3553.81)
1-Oct-99 ($ 2404.45)	14-Apr-00 ($ 3207.96)
3-Jun-00 ($ 3790.55)	14-Apr-00 ($ 3553.81)

Results are obtained under assumption that prices begin to fall down before the end of 2001 (this was really expected by most traders).

PROBLEM I. Let $U = U(x)$ be a utility function (e.g., $U(x) = \log x$ or $U(x) = x$). In the paper

A.Shiryaev, Z.Xu, X.Y.Zhou. Thou Shalt Buy and Hold

the following problem was considered:

To find an optimal stopping time τ^* such that

$$\mathsf{E}U\Big(\frac{P_{\tau^*}}{M_T}\Big) = \sup_{\tau \le T} \mathsf{E}U\Big(\frac{P_\tau}{M_T}\Big),$$

where $P_t = S_t/B_t$ is discounted price,

$$dB_t = rB_t\,dt, \quad B_0 = 1, \qquad dS_t = S_t(\mu\,dt + \sigma\,dW_t), \quad S_0 = 1.$$

Prices P_t solve the equation

$$dP_t = P_t((\mu - r)\,dt + \sigma\,dW_t), \qquad P_0 = 1,$$

and $P_t = \exp(\nu t + \sigma W_t)$, where $\nu = \mu - r - \sigma^2/2$.

THEOREM I. For the <u>linear</u> function $U(x) = x$ the optimal stopping time is **degenerate**:

$$\tau^* = \begin{cases} T, & \text{if } \nu > 0, \\ 0, & \text{if } \nu \le 0. \end{cases} \qquad (*)$$

(The case $0 < \nu$ and $\nu \le -\sigma^2/2$ was considered in the paper by A.Shiryaev, Z.Xu, X.Y.Zhou; the case $-\sigma^2/2 < \nu \le 0$ was studied by J. du Toit, G.Peskir.)

The case of the <u>logarithmic</u> function $U(x) = \log x$ is simple:

$$\sup_{\tau \le T} \mathsf{E} \log \frac{P_\tau}{M_T} = \sup_{\tau \le T} \mathsf{E}[\nu\tau + \sigma W_\tau - M_T] = \sup_{\tau \le T} \mathsf{E}[\nu\tau + \sigma W_\tau] - \mathsf{E}M_T$$

$$= \sup_{\tau \le T} \mathsf{E}\nu\tau - \mathsf{E}M_T = \begin{cases} \nu T - \mathsf{E}M_T, & \text{if } \nu > 0, \\ -\mathsf{E}M_T, & \text{if } \nu \le 0. \end{cases}$$

So, in the logarithmic case the optimal stopping time is given by $(*)$.

PROBLEM II. Now we consider the model

$$dS_t = S_t\Big[(\mu_1 I(t < \theta) + \mu_2 I(t \geq \theta))\, dt + \sigma\, dW_t\Big]$$

with $\mu_1 > \mu_2$, $\nu_1 \equiv \mu_1 - r - \sigma^2/2 > 0$, $\nu_2 \equiv \mu_2 - r - \sigma^2/2 < 0$ so that

$$\mu_2 - \tfrac{1}{2}\sigma^2 < r < \mu_1 - \tfrac{1}{2}\sigma^2.$$

If the value μ_1 remains unchanged on the whole interval $[0, T]$ and $\nu_1 > 0$, then by the previous result (Problem I) we should

hold the stock until time $t = T$ and sell it at this time.

But in fact the model admits that at a certain random time θ the regime switches from μ_1 to μ_2 and if $\nu_2 \equiv \mu_2 - r - \sigma^2/2 < 0$, then again by the previous problem we should

sell this stock at this time θ.

However, this time is unobservable and so the time of selling must depend on the "correct" estimation of the time θ.

Our second problem (Problem II) is the following:

To find "one-time rebalancing" stopping time τ_T^* such that

$$V_T = \sup_{\tau \le T} \mathsf{E} U\Big(\frac{P_\tau}{M_T}\Big), \qquad P_\tau = \frac{S_\tau}{B_\tau}.$$

We shall consider the case $U(x) = \log x$, i.e.,

$$V_T = \sup_{\tau \le T} \mathsf{E} \log \frac{P_\tau}{M_T} = \sup_{\tau \le T} \mathsf{E} \log P_\tau - \mathsf{E} \log M_T.$$

Assume that "hidden" parameter θ has an exponential distribution

$$\mathsf{P}(\theta = 0) = \pi, \quad \mathsf{P}(\theta > t \,|\, \theta > 0) = e^{-\lambda t},$$

where $\lambda > 0$ is known and $\pi \in [0, 1)$. Brownian motion W and θ in

$$dS_t = S_t\Big[(\mu_1 I(t < \theta) + \mu_2 I(t \ge \theta))\, dt + \sigma\, dW_t\Big]$$

are independent.

We see that $P_t = S_t/B_t = \exp X_t$, where

$$X_t = \int_0^t \nu(s,\theta)\,ds + \sigma W_t,$$

$$\nu(s,\theta) = \mu(s,\theta) - r - \frac{1}{2}\sigma^2, \quad \mu(s,\theta) = \mu_1 I(s < \theta) + \mu_2 I(s \geq \theta).$$

LEMMA 1. For any stopping time $\tau \leq T$ $(< \infty)$

$$\boxed{\mathsf{E}\log \mathsf{P}_\tau \equiv \mathsf{E}X_\tau = \mathsf{E}\int_0^\tau [\nu_1 - (\nu_1 - \nu_2)\pi_s]\,ds} \qquad (**)$$

where $\pi_s = \mathsf{P}(\theta \leq s \,|\, \mathcal{F}_s)$, $\mathcal{F}_s = \sigma(S_u,\ u \leq s)$.

Proof. For $X_t = \int_0^t \nu(s,\theta)\,ds + \sigma W_t$ we have an **innovation representation**

$$X_t = \int_0^t \mathsf{E}[\nu(s,\theta) \,|\, \mathcal{F}_s]\,ds + \sigma \overline{W}_t,$$

where $\overline{W} = (\overline{W}_t, \mathcal{F}_t)$ is an innovation (Wiener) process.

Since $\mathsf{E}[\nu(s,\theta) \,|\, \mathcal{F}_s] = \nu_1(1-\pi_s) + \nu_2\pi_s = \nu_1 - (\nu_1 - \nu_2)\pi_s$, we get the representation $(**)$.

LEMMA 2. For $(\pi_t)_{t\geq 0}$ we have

$$\boxed{d\pi_t = \lambda(1-\pi_t)\,dt + \frac{\nu_2-\nu_1}{\sigma}\,\pi_t(1-\pi_t)\,d\overline{W}_t}$$

where

$$\overline{W}_t = \frac{1}{\sigma}\Big[X_t - \int_0^t (\nu_1(1-\pi_s)+\nu_2\pi_s)\,ds\Big].$$

Proof is well known and can be done in the following way.

Define $\varphi_t = \dfrac{\pi_t}{1-\pi_t}$, $L_t = \dfrac{d\mathsf{P}_t^0}{d\mathsf{P}_t^\infty}$, where $\mathsf{P}_t^i = \mathsf{Law}(X_s,\ s \le t \,|\, \theta = i)$. Then

$$L_t = \exp\Big\{\frac{\nu_2-\nu_1}{\sigma^2}\,X_t - \frac{1}{2}\,\frac{\nu_2^2-\nu_1^2}{\sigma^2}\,t\Big\}, \qquad dL_t = L_t\,\frac{\nu_2-\nu_1}{\sigma^2}\,(dX_t - \nu_1\,dt).$$

By the Bayes formula,

$$\varphi_t = \varphi_0 e^{\lambda t}\frac{d\mathsf{P}_t^0}{d\mathsf{P}_t^\infty} + \lambda e^{\lambda t}\int_0^t e^{-\lambda s}\frac{d\mathsf{P}_t^s}{d\mathsf{P}_t^\infty}\,ds = \varphi_0 e^{\lambda t} L_t + \lambda e^{\lambda t}\int_0^t e^{-\lambda s}\frac{L_t}{L_s}\,ds,$$

where we used the property $\dfrac{d\mathsf{P}_t^s}{d\mathsf{P}_t^\infty} = \dfrac{L_t}{L_s}$.

By the Itô formula,

$$d\varphi_t = \Big[\lambda(1+\varphi_t) - \varphi_t\nu_1\,\frac{\nu_2-\nu_1}{\sigma^2}\Big]\,dt + \varphi_t\,\frac{\nu_2-\nu_1}{\sigma^2}\,dX_t$$

with $\varphi_0 = \pi/(1-\pi)$. From $\pi_t = \varphi_t/(1+\varphi_t)$ it follows

$$d\pi_t = (1-\pi_t)\Big[\lambda - \nu_1\,\frac{\nu_2-\nu_1}{\sigma^2}\,\pi_t - \frac{(\nu_2-\nu_1)^2}{\sigma^2}\,\pi_t^2\Big]\,dt + \frac{\nu_2-\nu_1}{\sigma^2}\pi_t(1-\pi_t)\,dX_t,$$

$$\text{where}\quad X_t = \int_0^t [\nu_1 - (\nu_1-\nu_2)\pi_s]\,ds + \sigma\overline{W}_t.$$

So, $d\pi_t = \lambda(1-\pi_t)\,dt + \dfrac{\nu_2-\nu_1}{\sigma^2}\pi_t(1-\pi_t)\,d\overline{W}_t$. Since

$$\mathsf{E}\log\mathsf{P}_\tau = \mathsf{E}X_\tau = \mathsf{E}\Big\{\int_0^\tau [\nu_1 - 1 - (\nu_1 - 1 - \nu_2)\pi_s]\,ds + \sigma\overline{W}_\tau\Big\}$$

and $\mathsf{E}\overline{W}_\tau = 0$ ($\tau \le T$), we get the representation

$$\mathsf{E}\log\mathsf{P}_\tau = \mathsf{E}\int_0^\tau [\nu_1 - (\nu_1-\nu_2)\pi_t]\,dt.$$

REMARK. For $P_t = e^{X_t}$ we obtain

$$\mathsf{E} P_t = \mathsf{E} \exp\Big\{\int_0^\tau [(\mu_1 - r) - (\mu_1 - \mu_2)\widehat{\pi}_s]\, ds\Big\},$$

where $(\widehat{\pi}_t)_{t \le T}$ has the stochastic differential

$$d\widehat{\pi}_t = (1 - \widehat{\pi}_t)[\lambda + \widehat{\pi}_t(\nu_2 - \nu_1)]\, dt + \frac{\nu_2 - \nu_1}{\sigma^2}\widehat{\pi}_t(1 - \widehat{\pi}_t)\, d\overline{W}_t.$$

Return to the problem of finding

$$V_T = \sup_{\tau \le T} \mathsf{E} \log P_\tau = \sup_{\tau \le T} \int_0^\tau [\nu_1 - (\nu_1 - \nu_2)\pi_t]\, dt.$$

LEMMA 3. For $V_T = V_T(\lambda;\pi)$ we have the representation

$$\boxed{V_T(\lambda;\pi) = \frac{\nu_1}{\lambda}\,(1-\pi) - \frac{\nu_1}{\lambda}\,R_T(c;\pi)}$$

where $c = \lambda|\nu_2|/\nu_1$ and

$$R_T(c;\pi) = \inf_{\tau\le T}\{\mathsf{P}(\tau\le\theta) + c\,\mathsf{E}(\tau-\theta)^+\}.$$

Proof. From $d\pi_t = \lambda(1-\pi_t)\,dt + \dfrac{\nu_2-\nu_1}{\sigma}\pi_t(1-\pi_t)\,d\overline{W}_t$ we find

$$\lambda t = (\pi_t - \pi) + \lambda\int_0^t \pi_s\,ds - \frac{\nu_2-\nu_1}{\sigma}\int_0^t \pi_s(1-\pi_s)\,d\overline{W}_s.$$

So, $\nu_1 t = \frac{\nu_1}{\lambda}(\pi_t - \pi) + \nu_1 \int_0^t \pi_s \, ds - \frac{\nu_1(\nu_2 - \nu_1)}{\lambda\sigma} \int_0^t \pi_s(1 - \pi_s) \, d\overline{W}_s$
and

$$\begin{aligned}
&\mathsf{E} \int_0^\tau [\nu_1 - (\nu_1 - \nu_2)\pi_t] \, dt \\
&\quad = \frac{\nu_1}{\lambda} \mathsf{E}\Big\{(\pi_\tau - \pi) + \frac{\nu_2}{\nu_1} \lambda \int_0^\tau \pi_t \, dt\Big\} \\
&\quad = -\frac{\nu_1}{\lambda}\pi - \frac{\nu_1}{\lambda} \mathsf{E}\Big\{\pi_\tau + \frac{|\nu_2|}{\nu_1} \lambda \int_0^\tau \pi_t \, dt\Big\} \\
&\quad = \frac{\nu_1}{\lambda}(1 - \pi) - \frac{\nu_1}{\lambda} \mathsf{E}\Big\{(1 - \pi_\tau) + \frac{|\nu_2|}{\nu_1} \lambda \int_0^\tau \pi_t \, dt\Big\}.
\end{aligned}$$

Note that $\mathsf{P}(\tau < \theta) = \mathsf{E}I(\tau < \theta) = \mathsf{E}\mathsf{E}(I(\tau < \theta) \,|\, \mathcal{F}_t^X) = \mathsf{E}(1 - \pi_\tau)$ and

$$\mathsf{E}(\tau - \theta)^+ = \mathsf{E}\int_0^T I(\theta \le s \le \tau)\,ds = \mathsf{E}\int_0^T \mathsf{E}[I(\theta \le s)I(s \le \tau) \,|\, \mathcal{F}_s^X]\,ds$$
$$= \mathsf{E}\int_0^T I(s \le \tau)\mathsf{E}[I(\theta \le s) \,|\, \mathcal{F}_s^X]\,ds = \mathsf{E}\int_0^\tau \pi_s\,ds.$$

So, $\mathsf{E}\Big\{(1 - \pi_\tau) + \dfrac{|\nu_2|}{\nu_1}\lambda \displaystyle\int_0^\tau \pi_t\,dt\Big\} = \mathsf{P}(\theta \le \tau) + \dfrac{|\nu_2|}{\nu_1}\lambda\mathsf{E}(\tau - \theta)^+$ and

$$\mathsf{E}\int_0^\tau [\nu_1 - (\nu_1 - \nu_2)\pi_t]\,dt = \frac{\nu_1}{\lambda}(1 - \pi) - \frac{\nu_1}{\lambda}\Big\{\mathsf{P}(\theta \le \tau) + \frac{|\nu_2|}{\nu_1}\lambda\mathsf{E}(\tau - \theta)^+\Big\}.$$

Taking infimum over $\tau \le T$, we find the required formula

$$V_T(\lambda; \pi) = \frac{\nu_1}{\lambda}(1 - \pi) - \frac{\nu_1}{\lambda}R_T(c; \pi).$$

The solution of the problem

$$R_T(c;\pi) = \inf_{\tau\le T}\Big\{\mathsf{P}(\tau\le\theta) + \frac{|\nu_2|}{\nu_1}\lambda\,\mathsf{E}(\tau-\theta)^+\Big\}.$$

for the case $T=\infty$ was obtained by the author: the optimal stopping time is given by

$$\tau^*_\infty = \inf\{t\ge 0\colon \pi_t \ge g^*_\infty\}, \qquad (\bullet)$$

with g^*_∞ a unique root of the equation $\Psi(g)=1$, where

$$\Psi(x) = \frac{c}{\rho}\int_0^x \exp\Big\{-\frac{\lambda}{\rho}[H(x)-H(y)]\Big\}\frac{dy}{y(1-y)^2}$$

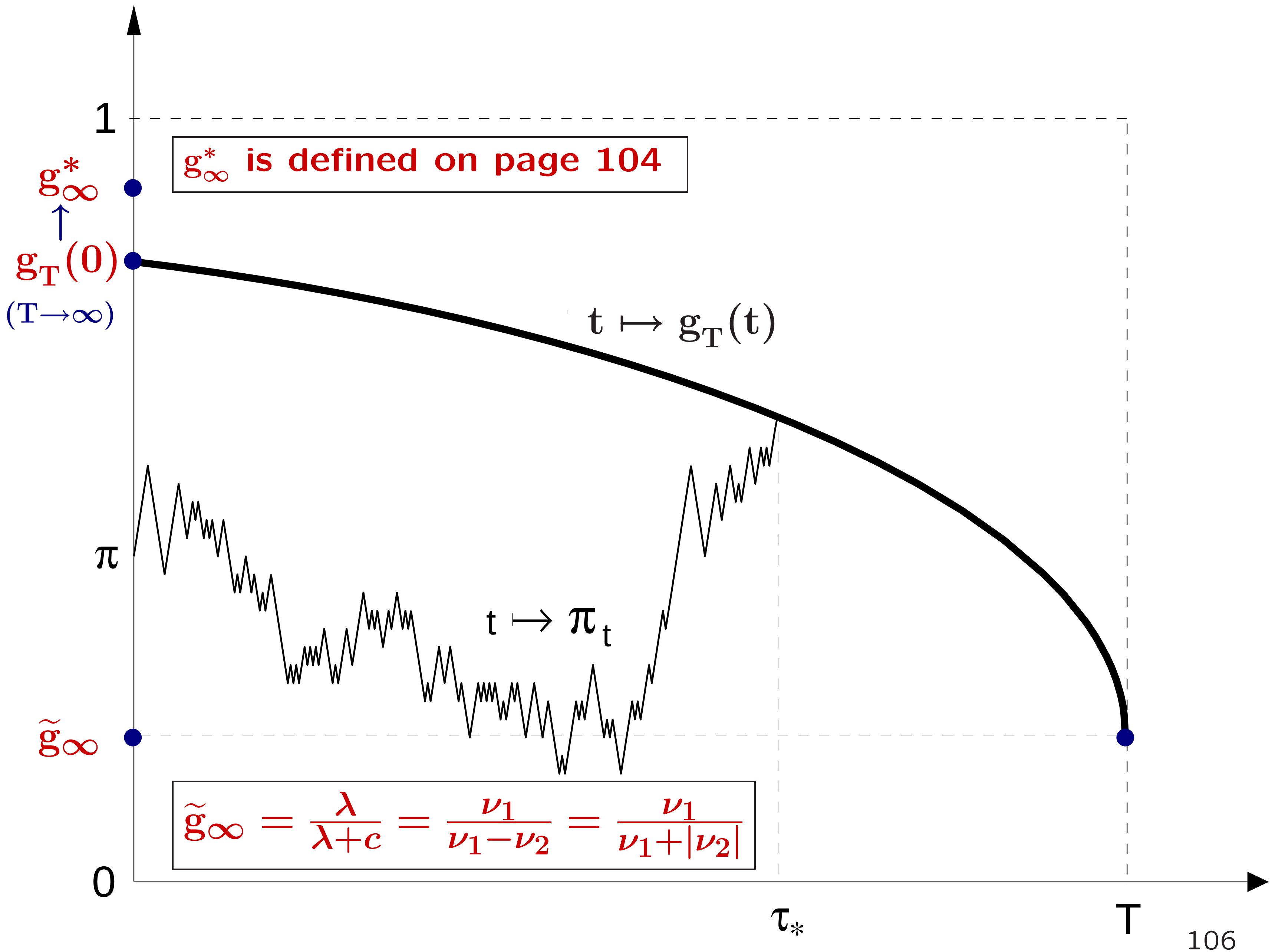
$$\text{with}\quad c = \frac{|\nu_2|}{\nu_1}\lambda, \quad \rho = \frac{(\mu_1-\mu_2)^2}{2\sigma^2}, \quad H(x) = \log\frac{x}{1-x} - \frac{1}{x}.$$

For the case $T < \infty$, the optimal stopping time is given by

$$\tau_T^* = \mathrm{inf}\{0 \le t \le T \colon \pi_t \ge g_T^*(t)\},$$

where $g^* = g_T^*(t)$, $0 \le t \le T$, is a unique solution of the nonlinear integral equation (Gapeev & Peskir)

$$\begin{aligned} \mathsf{E}_{t,g(t)}\pi_T = g(t) + c\int_0^{T-t} \mathsf{E}_{t,g(t)}[\pi_{t+u} I(\pi_{t+u} < g(t+u))]\,du \\ + \lambda \int_0^{T-t} \mathsf{E}_{t,g(t)}[(1-\pi_{t+u}) I(\pi_{t+u} < g(t+u))]\,du. \end{aligned}$$


1
g^*_∞ is defined on page 104
g^*_∞
$g_T(0)$
$(T \to \infty)$
$t \mapsto g_T(t)$
π
$t \mapsto \pi_t$
$\widetilde{g}_\infty$
$\widetilde{g}_\infty = \frac{\lambda}{\lambda + c} = \frac{\nu_1}{\nu_1 - \nu_2} = \frac{\nu_1}{\nu_1 + |\nu_2|}$
0
τ_*
T