Strongly taut immersions into Riemannian manifolds

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Introduction

Constructing the Morse function on a compact manifold M:

- Consider some immersion $f : M \longrightarrow \mathbb{R}^N$
- Consider the functions $L_q(x) = |x - q|^2$ on $\mathbb{R}^N$
- For generic $q \in \mathbb{R}^N$ $L_q \circ f$ is a Morse function on M

Weak Morse inequalities: $\mu_i \geqslant b_i = \dim_{\mathbb{Z}_2} H_i(M, \mathbb{Z}_2)$

If all equalities hold then the Morse function is called *perfect.*

Tautness in $\mathbb{R}^N$

When would $L_q \circ f$ be a perfect Morse function on M ?
The immersion $f: M \longrightarrow \mathbb{R}^N$ is called *taut*
if one of these equivalent conditions is satisfied:

Def.1. $L_q \circ f$ is a perfect Morse function on M for
generic $q \in \mathbb{R}^N$;
Def.2. For every closed Euclidean ball $B \subset \mathbb{R}^N$
the homomorphism
$i^* : H_*(f^{-1}(f(M) \cap B), \mathbb{Z}_2) \longrightarrow H_*(M, \mathbb{Z}_2)$
induced by inclusion map $i : f^{-1}(f(M) \cap B) \longrightarrow M$
is injective.

Actually, any taut immersion is an embedding.

Examples in $\mathbb{R}^2$

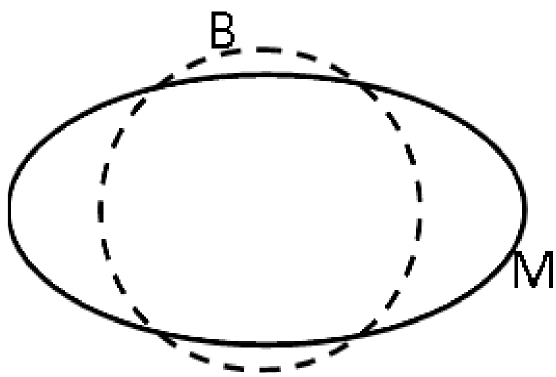

Fig. 1

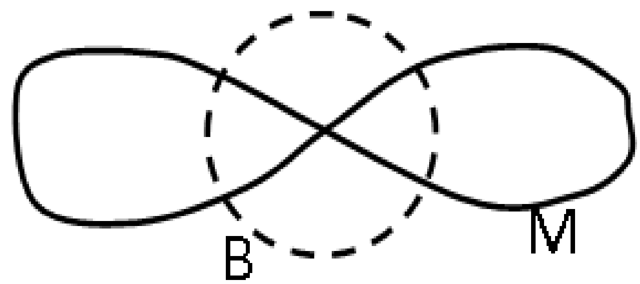

Fig. 2

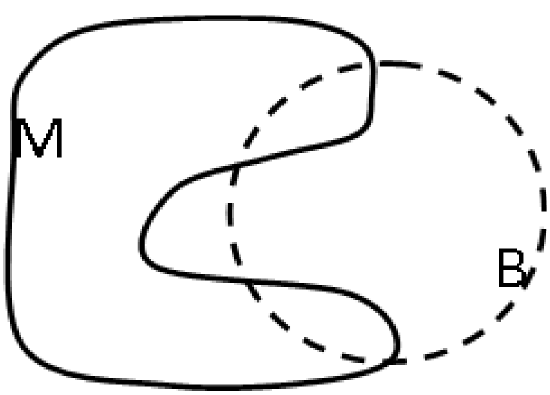

Fig. 3

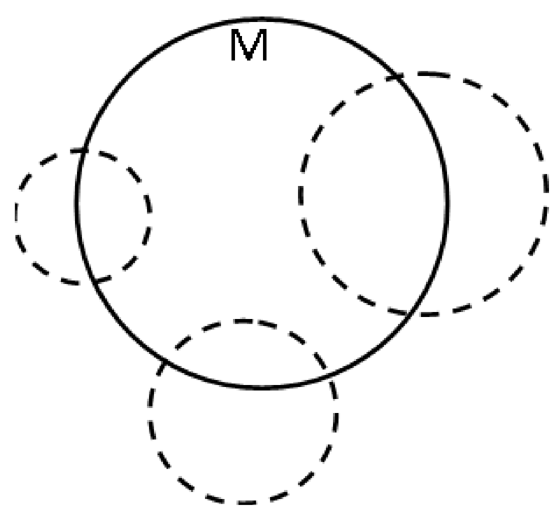

Fig. 4

Interesting facts

1. Sphere S^k can be tautly embedded into $\mathbb{R}^N$ only as a Euclidean sphere in the linear subspace.

2. Any tautly embedded submanifold is Dupin.

3. Only these 3-manifolds can be tautly embedded to some Euclidean space:

$$S^3,\ \mathbb{R}P^3,\ S^3/Q_8,\ S^1 \times S^2,\ S^1 \times \mathbb{R}P^2,\ (S^1 \times S^2)/\mathbb{Z}_2,\ T^3$$

Terng-Thorbergsson's generalization

C.-L. Terng, G. Thorbergsson
"Taut Immersions into Complete Riemannian Manifolds".

Let $f : M \longrightarrow N$ be an immersion into complete Riemannian manifold.

Let $P(p, f(M)) = \{\gamma : [0,1] \longrightarrow N | \gamma(0) = p, \gamma(1) \in f(M)\}$

Define $E_p : \gamma \mapsto \int_0^1 |\gamma'(t)|^2 \mathrm{d}t$

Def.3. The immersion f is called *taut,* if E_p is a perfect Morse functional on the space $P(p, f(M))$ for a generic point p.

Strong tautness

Def.4. We shall call an immersion $f : M \longrightarrow N$
into a complete Riemannian manifold *strongly taut,*
if the homomorphism
$i^* : H_*(f^{-1}(f(M) \cap B), \mathbb{Z}_2) \longrightarrow H_*(M, \mathbb{Z}_2)$
induced by the inclusion map, is injective for any closed metric ball $B \subset N$.

If the homomorphism is injective in k-dimensional homologies then the immersion is called *strongly k-taut.*

As in $\mathbb{R}^N$, any 0-taut immersion is an embedding.

An embedding f is 0-taut if and only if $f(M) \cap B$
is always connected.

Some results

Th.1. Let $M^{n-1} \hookrightarrow N^n$ be a strongly 0-taut embedding into the compact Riemannian manifold.

If the group $\pi_1(N^n, M^{n-1})$ is non-trivial then $N^n \backslash M^{n-1}$ is connected.

Sketch of proof.

$\gamma : [0,1] \longrightarrow N$ is the shortest path, which satisfies the condition that $\gamma \neq 1$ in $\pi_1(N^n, M^{n-1})$.

Consider $B(m, \frac{l(\gamma)}{2})$. Then $M \cap B = \{\gamma(0), \gamma(1)\}$.

So $\gamma(0) = \gamma(1)$. Now consider $\delta(t) = \gamma(1-t)$.

If $N \backslash M$ is not connected then $\delta(0) = \gamma(0)$ and $\delta'(0) = \gamma'(0)$ so $\gamma(t) = \delta(t)$,

which is impossible.

End of proof.

Illustrations for Th.1

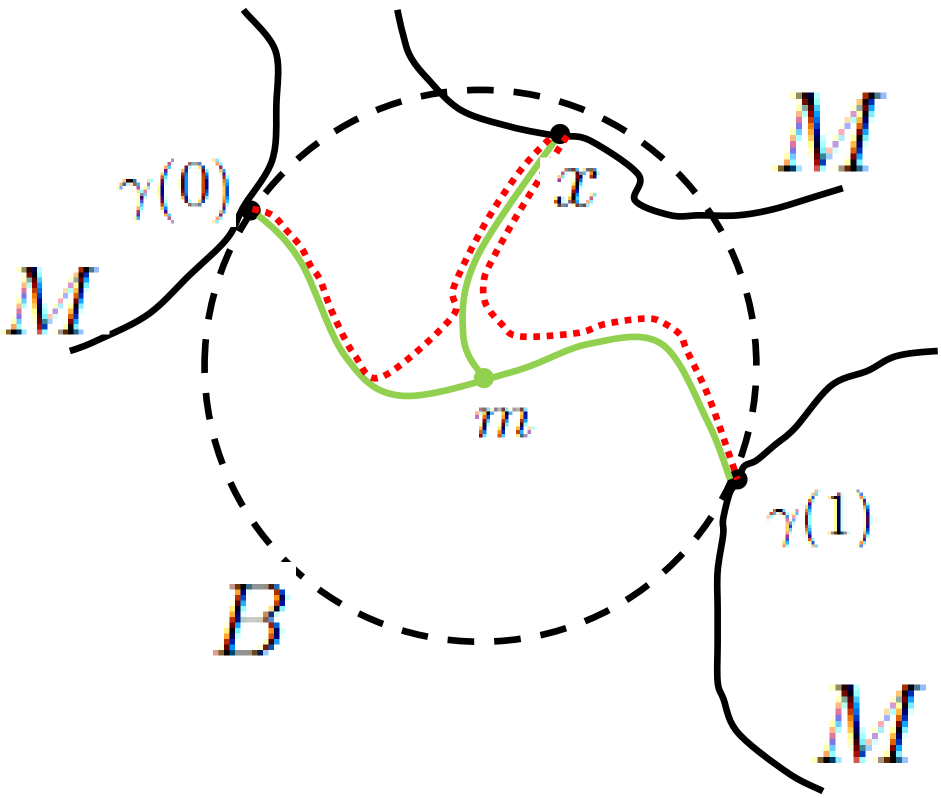

Fig. 5

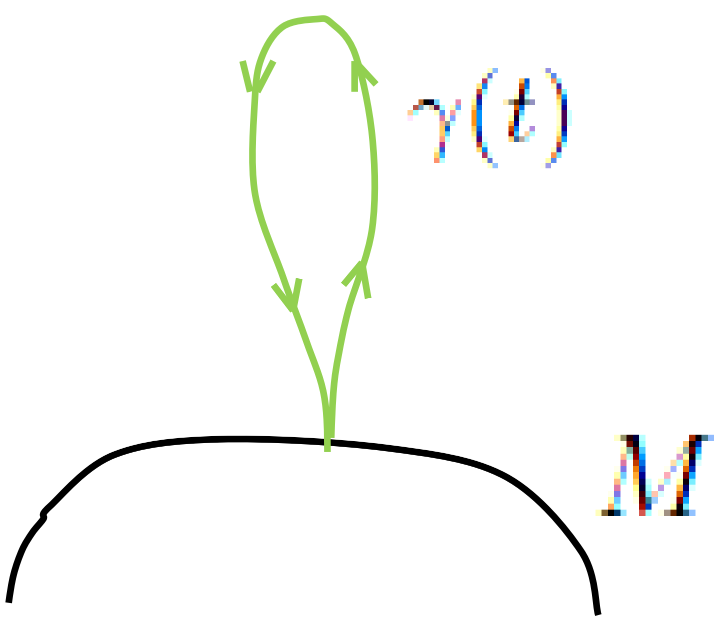

Fig. 6

Some results

Th.2. If $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a torus with the standard metric $g_{ij} = \delta_{ij}$ then $S^1 \hookrightarrow T^2$ is a strongly taut embedding if and only if S^1 is the line $x = Const$ or $y = Const$.

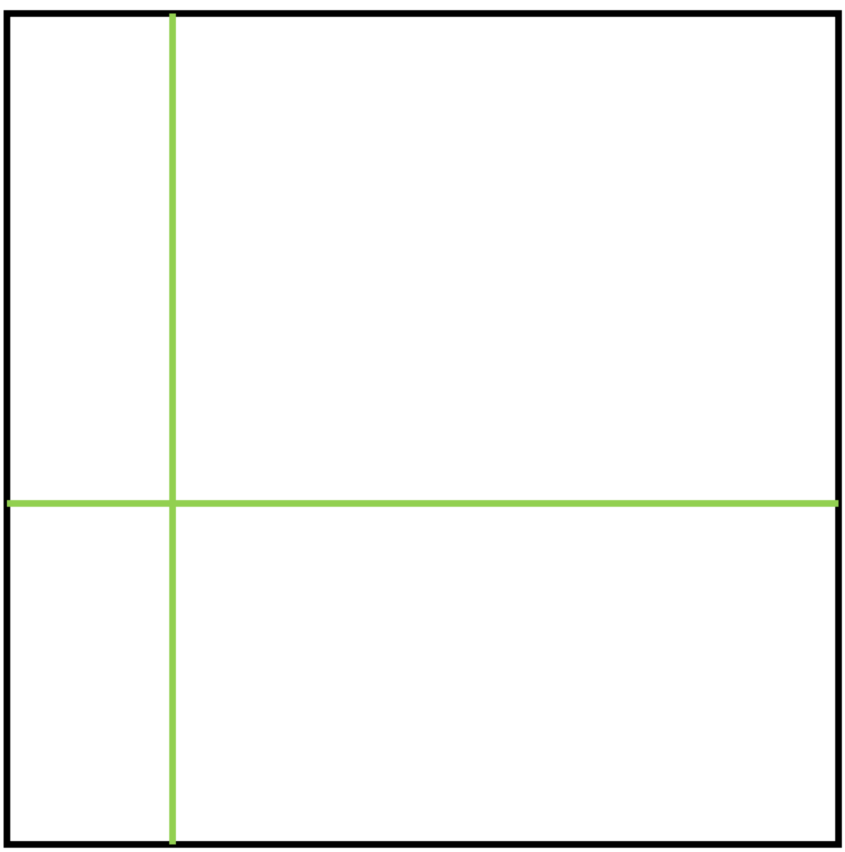

Fig. 7

Thanks for watching!