Elastic Curves in Riemannian Space

Adrian Spener

University of Ulm

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Agenda

Length of Curves

Let $(M, g =< \cdot, \cdot >)$ be a Riemannian manifold, $I \in \{[a,b], \mathbb{S}^1\}$, $\gamma : I \to M$ smooth. $\dot\gamma = T\gamma(\frac{\mathrm{d}}{\mathrm{d}t})$ is the push-forward of the vector field $\frac{\mathrm{d}}{\mathrm{d}t}$, i.e. the tangent vector field in $\mathcal{T}(\gamma)$.

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Definition

The length of γ is given by

$$\mathcal{L}(\gamma) := \int_I |\dot\gamma(t)|\,\mathrm{d}t = \int_I g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))^{\frac{1}{2}}\,\mathrm{d}t.$$

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Further assume that γ is an immersion, i.e. $|\dot\gamma| \neq 0$ on I), and γ is parametrized s.t. $|\dot\gamma| = 1$. Since M is Riemannian there exists a unique connection ∇ yielding D_t, the covariant derivative along γ.

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Definition

The curvature vector of γ is defined by $D_t\dot{\gamma} \in \mathcal{T}(\gamma)$.

Extremize $\mathcal{L}$

Let $W \in \mathcal{T}(\gamma)$ be a smooth vector field along γ.
The geodesic flow then yields the existence of a smooth variation $\gamma_\varepsilon(\cdot) : I \to M$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\gamma_\varepsilon\Big|_{\varepsilon=0} = W \text{ and } \gamma_0 = \gamma.$$

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Thus for the first variation of $\mathcal{L}$ we have by compatibility

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{L}(\gamma_\varepsilon)\Big|_{\varepsilon=0} = [< W, \dot{\gamma} >]_0^{\mathcal{L}(\gamma)} - 2\int_0^{\mathcal{L}(\gamma)} < D_t\dot{\gamma}, W > \mathrm{d}s$$

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where $s \in [0, \mathcal{L}(\gamma)]$ denotes the arc length, i.e. $\mathrm{d}s = \|\dot{\gamma}\|\,\mathrm{d}t$. Hence we see for fixed end points ($W|_{\partial I} \equiv 0$):

Proposition

The curve γ is critical (i.e. $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{L}(\gamma_\varepsilon)\big|_{\varepsilon=0} = 0\ \forall\ W \in \mathcal{T}(\gamma)$)
$\Leftrightarrow D_t\dot{\gamma} = 0$, i.e. if γ is geodesic.

Elastic Energy

Let $\gamma : I \to M$ be an immersion.

Definition

The Elastic Energy (or Bending Energy) of γ is defined as

$$\mathcal{E}(\gamma) := \int_0^{\mathcal{L}(\gamma)} |D_t \dot{\gamma}|^2 \, \mathrm{d}s$$

For any $\lambda > 0$, an Elastica (or Elastic Curve) is an immersion $\gamma : I \to M$ which is critical for $\mathcal{E}(\cdot) + \lambda \mathcal{L}(\cdot) =: \mathcal{E}^{\lambda}(\cdot)$.

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Definition

For a vector field $X \in \mathcal{T}(\gamma)$ we denote the normal component of $D_t\dot{\gamma}$ by

$$\nabla^\perp X := D_t X - \langle D_t X, \dot{\gamma}\rangle\,\dot{\gamma}.$$

With help of this abbreviation we can now formulate the first variation of $\mathcal{E}^\lambda$ in the next proposition.

The First Variation

Proposition

Let γ_ε be a smooth variation of γ with variation field W. Then the following equation holds:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}^\lambda(\gamma_\varepsilon)\Big|_{\varepsilon=0}$$

where R is the Riemannian curvature endomorphism.

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$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{E}^\lambda(\gamma_\varepsilon)\Big|_{\varepsilon=0} = \int_0^{\mathcal{L}(\gamma)} \left\langle 2R(D_t\dot\gamma,\dot\gamma)\dot\gamma + 2\left(\nabla^\perp\right)^2 D_t\dot\gamma + (|D_t\dot\gamma|^2 - \lambda)D_t\dot\gamma, W \right\rangle \mathrm{d}s$$

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$$+\left[2\left\langle\nabla^\perp W, D_t\dot\gamma\right\rangle - \left\langle 2\nabla^\perp D_t\dot\gamma + (|D_t\dot\gamma|^2-\lambda)\dot\gamma, W\right\rangle\right]_0^{\mathcal{L}(\gamma)}.$$

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where R is the Riemannian curvature endomorphism. For closed curves $\gamma : \mathbb{S}^1 \to M$ the boundary term vanishes and thus the Euler-Lagrange Equation is given by

$$2R(D_t\dot\gamma, T)T + 2\left(\nabla^\perp\right)^2 D_t\dot\gamma + (|D_t\dot\gamma|^2-\lambda)D_t\dot\gamma = 0 \text{ on } \gamma. \tag{1}$$

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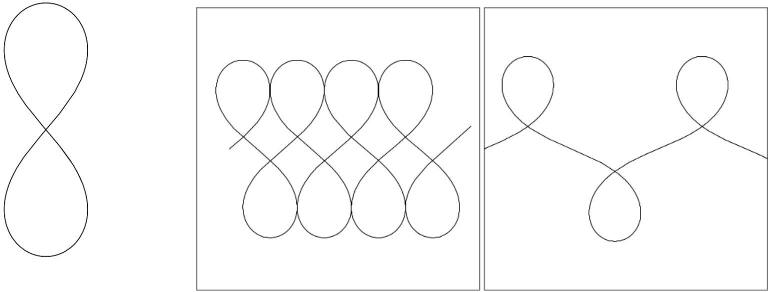
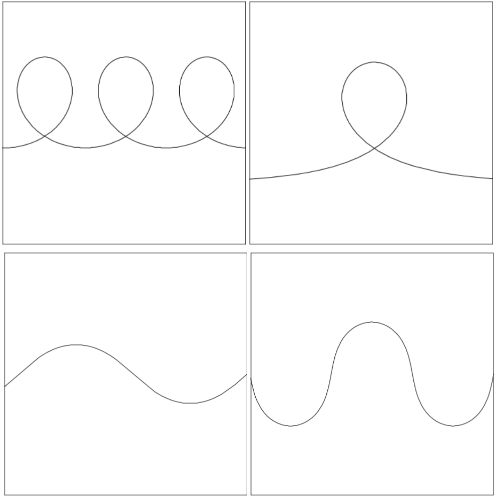
For space forms, where the sectional curvature is constant to a real number K_0 the Euler-Lagrange Equation (1) simplifies to

$$2\left(\nabla^\perp\right)^2 D_t\dot\gamma + (|D_t\dot\gamma|^2-\lambda+2K_0)D_t\dot\gamma = 0 \text{ on } \gamma.$$



(Taken from [1])



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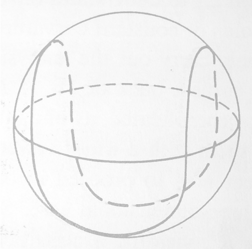

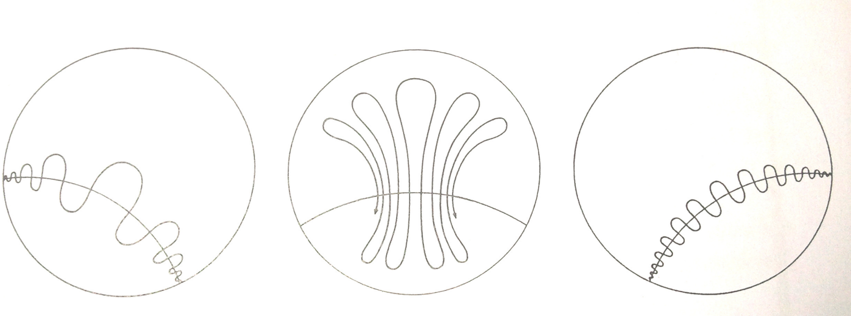

(Taken from [2])

Willmore Surfaces

Let $\iota : \Sigma^2 \hookrightarrow \mathbb{R}^3$ be an immersion, Σ^2 a two-dimensional compact manifold.

Definition

Σ is called Willmore Surface if it is critical for

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 \, \mathrm{dVol}_g$$

where $g = \iota^* g_{Eucl}$ is the induced metric and $H = \frac{1}{2}\,\mathrm{tr}(S) = \frac{1}{2}(\kappa_1 + \kappa_2)$ is the mean curvature of Σ.

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Proposition

The Euler-Lagrange equation for $\mathcal{W}$ is given by

$$\Delta H + 2H(H^2 - K) \text{ on } \Sigma \tag{2}$$

Proposition (Langer, Singer '84 [3])

Let γ be a closed curve in the hyperbolic half-plane. Then for the torus of revolution Γ of γ in $\mathbb{R}^3$ we have $\mathcal{W}(\Gamma) = \frac{\pi}{2}\mathcal{E}_{\mathbb{H}}(\gamma)$.

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Proposition (Dall'Acqua, Deckelnick, Grunau '08 [4])

For a graph $u : [-1,1] \to \mathbb{R}^+$ smooth consider the surface of revolution given by

$$\Gamma := \left\{ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u(x) \\ 0 \\ x \end{pmatrix}, \phi \in \mathbb{R}, x \in [-1,1] \right\}$$

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Then

$$\mathcal{W}(\Gamma) = \frac{\pi}{2}\mathcal{E}^0_{\mathbb{H}}\left(\begin{pmatrix} u(\cdot) \\ \cdot \end{pmatrix}\right) - 2\pi\left[\frac{\dot{u}(t)}{\sqrt{1+\dot{u}^2(t)}}\right]_{-1}^{1}$$

where $\mathcal{E}_{\mathbb{H}}$ denotes the elastic energy of the curve $[-1,1] \ni t \mapsto (u(t), t) \in \mathbb{H}$ in the hyperbolic half-plane.

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