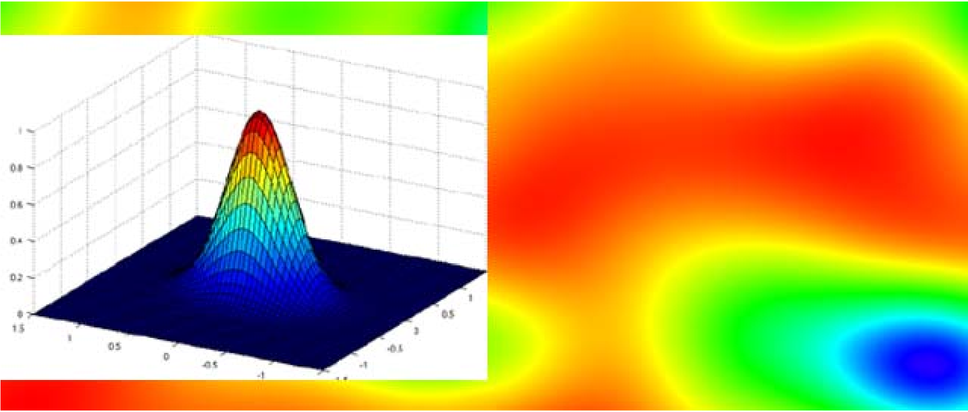

Prediction of Stable Stochastic Processes

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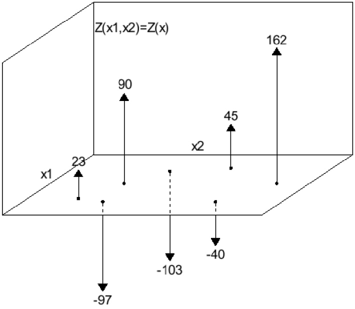
Part 1: Introduction

- Stochastic prediction
- Applications
- Stationary random fields
- Kriging

Spatial data



$\{X(t_i)\}_{i=1}^n$ - spatial data in observation window $W \subseteq \mathbb{R}^d$. They are interpreted as a realisation of a real–valued random field

$$X = \{X(t) : \ t \in \mathbb{R}^d\}$$

which is a spatially indexed family of random variables defined on a joint probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Stochastic prediction

Let the observations $X(t_1), \ldots, X(t_n)$ of a random field $X = \{X(t),\ t \in \mathbb{R}^d\}$ be given for $t_1, \ldots t_n \in W$, $W \subset \mathbb{R}^d$ being a compact set.

Find a predictor $\widehat{X}(t)$ for $X(t)$, $t \notin \{t_1, \ldots, t_n\}$ that is optimal in some sense and has a number of nice properties such as exactness, continuity, etc.

Examples of stochastic prediction methods

- Kriging
- Geoadditive regression models
- Whittaker smoothing
- Randomly coloured mosaics
- ...

Prediction of wide sense stationary random functions

- Mathematical foundations: extrapolation of stationary time series A.N. Kolmogorov (1941), N.Wiener (1949).
- Origins of geostatistics: D. Krige (1951), B. Mathérn (1960), L. Gandin (1963), G. Matheron (1962-63).

Stationary random fields

Random field $X = \{X(t) : t \in \mathbb{R}^d\}$ is (strictly) stationary if its probability law is translation invariant, i.e., all finite dimensional distributions are invariant with respect to any shifts in $\mathbb{R}^d$: for all $h \in \mathbb{R}^d$, $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}^d$ holds

$$(X(t_1 + h), \ldots, X(t_n + h)) \stackrel{d}{=} (X(t_1), \ldots, X(t_n)).$$

Random field $X = \{X(t) : t \in \mathbb{R}^d\}$ is stationary of 2nd order if $E\,X^2(t) < \infty$ for all $t \in \mathbb{R}^d$ and

- $E(X(t)) = \mu$ for all t.
- $\gamma(h) = \frac{1}{2} E\left[(X(t+h) - X(t))^2\right]$ depends only on vector h, but not on t.

Stationary random fields

- Strict stationarity $\not\Leftarrow\ \not\Rightarrow$ stationarity of second order
- A second order stationary random field is called isotropic if $C(h) = C(|h|)$, $h \in \mathbb{R}^d$.

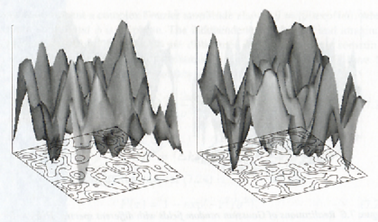
Correlation structure:

Let the random field $X = \{X(t)\}$ be stationary of second order.

- Variogram: $\gamma(h) = \frac{1}{2} E\left[(X(t+h) - X(t))^2\right]$
- Covariance function: $C(h) = E\left[X(t) \cdot X(t+h)\right] - \mu^2$
- $\gamma(h) = C(0) - C(h)$

Example: Gaussian random fields

- A random field $\{X(t)\}$ is called Gaussian if the distribution of $(X(t_1), ..., X(t_n))^\top$ is multivariate Gaussian for each $1 \leq n < \infty$ and $t_1, ..., t_n \in \mathbb{R}^d$.



The distribution of X is completely defined by the mean value function $\mu(t) = E\,X(t)$ and covariance function $C(s,t) = \mathrm{Cov}\big(X(s), X(t)\big)$, $s, t \in \mathbb{R}^d$. Hence: strict stationarity $\Longleftrightarrow$ stationarity of second order.

Ordinary Kriging (D. Krige (1951), G. Matheron (1962-63))

- Assumptions: X is stationary of second order.
- Notation

t_i	: locations of the sample points
$X(t_i)$	: observed values of X
n	: number of sample points
λ_i	: weights

- Estimator: $\widehat{X}(t) = \sum_{i=1}^{n} \lambda_i X(t_i)$, where $\sum_{i=1}^{n} \lambda_i = 1$.
- The weights λ_i are chosen such that the estimation variance $\sigma_E^2 = Var(\widehat{X}(t) - X(t))$ is minimized.

Ordinary Kriging

- $\widehat{X}(t)$ is unbiased: $E\,\widehat{X}(t) = \mu$ since $\sum_{i=1}^{n} \lambda_i = 1$
- $\sigma_E^2 \to \min = \sigma_{OK}^2$: solve the Lagrange equations

$$\begin{cases} \sum_{j=1}^{n} \lambda_j \gamma(t_j - t_i) + \nu = \gamma(t - t_i), \quad i = 1, \ldots, n, \\ \sum_{j=1}^{n} \lambda_j = 1. \end{cases}$$

- The minimal estimation variance:

$$\sigma_{OK}^2 = \nu + \sum_{i=1}^{n} \lambda_i \gamma(t_i - t)$$

Ordinary Kriging

Variogram fitting

To find the weights λ_i from the system of linear equations, the variogram $\gamma(h)$ has to be known or estimated from the data $X(t_1), \ldots, X(t_n)$.

- Matheron's estimator:

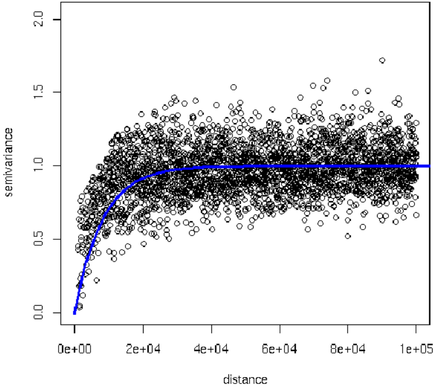
$$\hat{\gamma}(h) = \frac{1}{2N(h)} \sum_{i,j: t_i - t_j \approx h} \left(X(t_i) - X(t_j)\right)^2,$$

  $N(h)$ is the number of pairs $(t_i, t_j) : t_i - t_j \approx h$. Computations are made for h on a grid in $\mathbb{R}^d$.

- $\hat{\gamma}(h)$ not conditionally negative definite $\Rightarrow$ a valid variogram model has to be fitted to $\hat{\gamma}(h)$ e.g. by least squares

Variogram fitting

Variogram point cloud and a fitted exponential variogram



Properties of ordinary kriging

- The kriging predictor exists and is unique.
- BLUE: best linear unbiased estimator by definition.
- Exactness: $\widehat{X}(t_i) = X(t_i)$ a.s., $i = 1, \ldots, n$
- If X is a stationary Gaussian random field, $X(t) \sim N(\mu, \sigma^2)$, then $\widehat{X}$ is Gaussian as well, and $\widehat{X}(t) \sim N(\mu, \sigma_0^2(t))$ with

$$\sigma_0^2(t) = \sigma^2 + \nu - \sum_{i=1}^{n} \lambda_i \gamma(t_i - t)$$

Part 2: Stable laws and integration

- Motivation
- Stable distributions
- Covariation
- Random measures
- Stochastic integration

Random fields without a finite second moment

- Before: Extrapolation of 2nd order stationary random fields
- Now: need more flexible models and corresponding extrapolation methods for random fields with infinite variance. Why do we take care?

Motivation

Natural disasters and their mapping (geosciences)

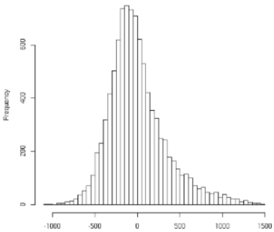
Hundred year flood, 2002

Winter storm “Kyrill”, 2007

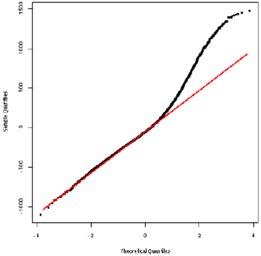
Motivation: storm insurance in Austria

Centers of 2047 postal code regions in Austria

Motivation: storm insurance in Austria



Histogram of the deviations



Q-Q plot of the deviations

- Goal: Spatial modelling of the deviations $Y(t) = X(t) - \mu(t)$ from the mean claim payments $\mu(t) = E\,X(t)$ with random fields.
- However, the distribution of the deviations is not Gaussian (rather skewed and heavy-tailed).

Stable distributions

Stable distributions (Kchinchine, Levy, 1930s):

- A random variable X is said to have a stable distribution if there is a sequence of i.i.d. random variables $Y_1, Y_2, \ldots$ and sequences of positive numbers $\{d_n\}$ and real numbers $\{a_n\}$, such that

  $$\frac{Y_1 + \ldots + Y_n}{d_n} + a_n \xrightarrow{d} X$$

  where $\xrightarrow{d}$ denotes convergence in distribution.

Stable distributions

- A random variable X is stable if and only if for $A, B > 0 \; \exists C > 0, D \in \mathbb{R}$:

  $$AX_1 + BX_2 \stackrel{d}{=} CX + D$$

  where X_1 and X_2 are independent copies of X.

- There exists a number $\alpha \in (0, 2]$ (index of stability)

  such that $C^\alpha = A^\alpha + B^\alpha$

- Also referred to as $(\alpha-)$stable distribution

- For $\alpha = 2$: normal distribution

Stable distributions

- Characteristic function of an α-stable random variable $X \sim S_\alpha(\sigma, \beta, \mu)$, $0 < \alpha \leq 2$:

$$E\left(e^{i\theta X}\right) = \begin{cases} e^{-\sigma^\alpha |\theta|^\alpha (1 - i\beta \operatorname{sgn}\theta \tan \frac{\pi\alpha}{2}) + i\mu\theta} & \text{if } \alpha \neq 1 \\ e^{-\sigma|\theta|(1 + i\beta \frac{2}{\pi} \operatorname{sgn}\theta \ln|\theta|) + i\mu\theta} & \text{if } \alpha = 1 \end{cases}$$

- Parameters σ, β, μ are unique for $\alpha \in (0, 2)$:
  - $\mu \in \mathbb{R}$: shift
  - $\beta \in [-1, 1]$: skewness (form) , $\beta = 0$: symmetry
  - $\sigma \geq 0$: scale

Multivariate stable distributions

- A random vector $\boldsymbol{X} = (X_1, ..., X_d)^\top$ is called stable if for $A, B > 0 \; \exists C > 0, \boldsymbol{D} \in \mathbb{R}^d$:

$$A\boldsymbol{X}^{(1)} + B\boldsymbol{X}^{(2)} \stackrel{d}{=} C\boldsymbol{X} + \boldsymbol{D}$$

- Symmetric random vector
  A random vector $\boldsymbol{X}$ in $\mathbb{R}^d$ is *symmetric* if $\mathbb{P}(\boldsymbol{X} \in A) = \mathbb{P}(-\boldsymbol{X} \in A)$ for any Borel set $A \in \mathbb{R}^d$.

Multivariate stable distributions

- Characteristic function of an α-stable random vector $\boldsymbol{X} = (X_1, ..., X_d)^\top$, $0 < \alpha \le 2$:

$$E\left(e^{i\cdot\boldsymbol{\theta}^T\boldsymbol{X}}\right) = \begin{cases} e^{-\int_{S_d} |\boldsymbol{\theta}^T\boldsymbol{s}|^\alpha\left(1 - i\,\mathrm{sgn}\boldsymbol{\theta}^T\boldsymbol{s}\tan\frac{\pi\alpha}{2}\right)\Gamma(d\boldsymbol{s}) + i\boldsymbol{\theta}^T\boldsymbol{\mu}} & \text{if } \alpha \neq 1 \\ e^{-\int_{S_d} |\boldsymbol{\theta}^T\boldsymbol{s}|\left(1 + i\frac{2}{\pi}\mathrm{sgn}\boldsymbol{\theta}^T\boldsymbol{s}\ln|\boldsymbol{\theta}^T\boldsymbol{s}|\right)\Gamma(d\boldsymbol{s}) + i\boldsymbol{\theta}^T\boldsymbol{\mu}} & \text{if } \alpha = 1 \end{cases}$$

  where Γ is a finite (spectral) measure on the unit sphere S_d of $\mathbb{R}^d$ and $\mu \in \mathbb{R}^d$.
- Parameters Γ and μ are unique for $\alpha \in (0, 2)$:
  - $\mu \in \mathbb{R}$: shift
  - Γ: skewness (form) and scale together.

Stable distributions

- Symmetric stable random vector
  A symmetric α-stable random vector $\boldsymbol{X}$ ($S\alpha S$) in $\mathbb{R}^d$ has a characteristic function

$$\varphi_{\boldsymbol{X}}(\boldsymbol{\theta}) = e^{-\int_{S_d} |\langle \boldsymbol{\theta}, \boldsymbol{s} \rangle|^\alpha \Gamma(d\boldsymbol{s})}, \quad \boldsymbol{\theta} \in \mathbb{R}^d$$

  where the spectral measure Γ is symmetric on S_d.
- If Γ is not concentrated on a great sub-sphere of S_d, then $\boldsymbol{X}$ is called full-dimensional.

Stable distributions

Properties and characteristics

- Moments: if $p < \alpha$ then $E\,|X|^p < \infty$. For $p \geq \alpha$, it holds $E\,|X|^p = \infty$.
- Covariation: for an α-stable random vector $(X_1, X_2)^\top$, $1 < \alpha \leq 2$ with spectral measure Γ define

$$[X_1, X_2]_\alpha = \int_{S_1} s_1 s_2^{<\alpha-1>}\,\Gamma(ds_1, ds_2)$$

  where $a^{<p>} := |a|^p \mathrm{sgn}(a)$ for $a \in \mathbb{R}$ and $p \geq 0$.
- Gaussian case $\alpha = 2$: if $(X_1, X_2)^\top$ is a centered Gaussian random vector then

$$[X_1, X_2]_2 = \frac{1}{2}\mathrm{Cov}(X_1, X_2).$$

Covariation and moments

Lemma (Karcher, Shmileva, S. (2013))

Let $1 < \alpha < 2$ and suppose that $(X, Y)^{\mathsf{T}}$ is an α-stable random vector with spectral measure Γ such that $X \sim S_\alpha(\sigma_X, \beta_X, 0)$ and $Y \sim S_\alpha(\sigma_Y, \beta_Y, 0)$. For $1 \le p < \alpha$, it holds

$$\frac{\mathbb{E}\left(XY^{<p-1>}\right)}{\mathbb{E}|Y|^p} = \frac{[X, Y]_\alpha(1 - c \cdot \beta_Y) + c \cdot (X, Y)_\alpha}{\sigma_Y^\alpha},$$

where $(X, Y)_\alpha := \int_{S_1} s_1 |s_2|^{\alpha-1} \Gamma(ds)$ and $c := c_{\alpha,p}(\beta_Y)$ is a constant. If Y is symmetric, i. e. $\beta_Y = 0$, then $c = 0$.

Random measures

Let $(E, \mathcal{E}, m)$ be a measurable space with a σ–finite measure m, $\mathcal{E}_0 = \{A \in \mathcal{E} : \ m(A) < \infty\}$,

$$L_0(\Omega) = \{\text{random variables on } (\Omega, \mathcal{F}, P)\}.$$

An independently scattered stable random measure M with control measure m and skewness intensity $\beta : E \to \mathbb{R}$ is a random measure with independent α–stable increments, i.e., an a.s. σ–additive function $M : \mathcal{E}_0 \to L_0(\Omega)$ with

$$M(A) \sim S_\alpha\left((m(A))^{1/\alpha}, \int_A \beta(x)\, m(dx)/m(A), 0 \right)$$

for any $A \in \mathcal{E}_0$.

Stochastic integration

For $f \in L^\alpha(E)$, construct $I(f) = \int_E f(x)\, M(dx)$, where M is an independently scattered α-stable random measure on $(E, \mathcal{E})$ with control measure m and skewness intensity β.

- Simple functions: for $f(x) = \sum_{j=1}^n c_j \mathbb{1}(x \in A_j)$, $x \in E$, with $A_j \in \mathcal{E}_0$: $A_i \cap A_j \neq \emptyset$, $i \neq j$, we set

$$I(f) = \sum_{j=1}^{n} c_j M(A_j).$$

- General functions: for any $f \in L^\alpha(E)$, there exists a sequence of simple functions $f_n \uparrow f$ a.e. on E. Set

$$I(f) = p-\lim_{n \to \infty} I(f_n).$$

Stochastic integral

This limit exists and does not depend on the choice of the sequence $\{f_n\}$ tending to f.

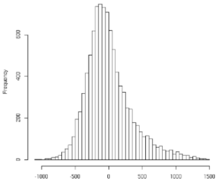
- Distribution: $I(f) \sim S_\alpha(\sigma_f, \beta_f, \mu_f)$, where

$$\sigma_f = \left(\int_E |f(x)|^\alpha \, m(dx) \right)^{1/\alpha} = \|f\|_{L^\alpha},$$

$$\beta_f = \frac{\int_E f(x)^{<\alpha>} \beta(x) \, m(dx)}{\int_E |f(x)|^\alpha \beta(x) \, m(dx)},$$

$$\mu_f = \begin{cases} 0, & \alpha \neq 1, \\ -\frac{2}{\pi} \int_E f(x) \beta(x) \log |f(x)| \, m(dx), & \alpha = 1. \end{cases}$$

Example: Spatial modelling of storm data (Austria)



Parameter estimation for the one-dimensional case:

The field X of deviations from the mean claim sizes has the univariate distribution $X(t) \sim S_\alpha(\sigma, \beta, \mu)$ with

α	β	σ	μ
1.3562	0.2796	234.286	6.7787

Part 3: Stable random fields

- Definition
- Spectral representation
- Examples
  - Subgaussian random fields
  - Stable motion
- Their properties

Stable random fields

- A random field $\{X(t),\ t \in \mathbb{R}^d\}$ is called α-stable if the distribution of $(X(t_1), ..., X(t_n))^\top$ is multivariate α–stable for any $1 \leq n < \infty$ and $t_1, ..., t_n \in \mathbb{R}^d$.
- A random field $\{X(t),\ t \in \mathbb{R}^d\}$ is called separable if $\exists$ a countable subset $T_0 \subset \mathbb{R}^d$ s.t. for all $t \in \mathbb{R}^d$ $X(t) = p - \lim_{k\to\infty} X(t_k)$ with $\{t_k,\ k \in \mathbb{N}\} \subset T_0$.
- Consider a stable random field

$$X(t) = \int_E f_t(x) M(dx),\ t \in \mathbb{R}^d,$$

where $f_t \in L^\alpha(E)$, $t \in \mathbb{R}^d$, and M is an α-stable independently scattered random measure with control measure m and skewness β.

Stable random fields

Spectral representation: for separable in probability α-stable fields with $0 < \alpha \leq 2$, $\alpha \neq 1$ it holds

$$\{X(t),\ t \in \mathbb{R}^d\} \stackrel{d}{=} \left\{ \int_0^1 f_t(x) M(dx) + \mu(t),\ t \in \mathbb{R}^d \right\}$$

where

- $f_t \in L^\alpha(0,1)$ for all $t \in \mathbb{R}^d$,
- M is an α-stable independently scattered random measure on $(0,1)$ with Lebesgue control measure and skewness intensity $\beta(x) = 1$, $x \in (0,1)$,
- $\mu : \mathbb{R}^d \to \mathbb{R}$ is some function.

Case $\alpha = 1$: open.

Examples: α–stable random fields

Sub-Gaussian random fields:

- Let $A \sim S_{\alpha/2}((\cos(\pi\alpha/4))^{2/\alpha}, 1, 0)$ and let $G = \{G(t), t \in \mathbb{R}^d\}$ be a stationary zero mean Gaussian random field with covariance function C. Assume that A is independent of G. The $S\alpha S$ random field $X = \{X(t),\ t \in \mathbb{R}^d\}$ with $X(t) = A^{1/2}G(t),\ t \in \mathbb{R}^d$ is called sub-Gaussian.
- Characteristic function of $X_{t_1,\ldots,t_n} = (X(t_1), \ldots, X(t_n))^\top$: for any $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}^d$ it holds

$$\varphi_{X_{t_1,\ldots,t_n}}(s_1, \ldots, s_n) = \exp\left\{ -\frac{1}{2} \left| \sum_{i,j=1}^{n} C(t_i - t_j)s_i s_j \right|^{\alpha/2} \right\}.$$

Examples: Simulation

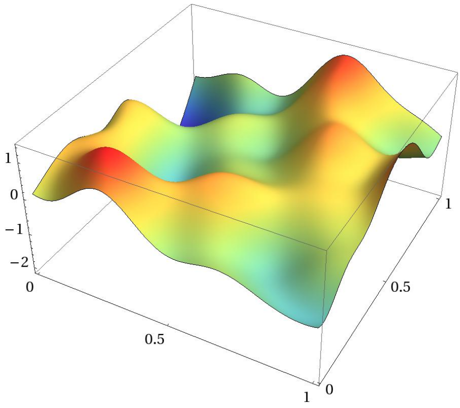
Subgaussian random field

$$X = \{A^{1/2} G(t), t \in [0,1]^2\}$$

with $\alpha = 1.5$, $A \sim S_{\alpha/2}((\cos(\pi\alpha/4))^{2/\alpha}, 1, 0)$ and G being a stationary isotropic Gaussian random field with covariance function

$$C(h) = 7 \exp\{-(h/0.1)^2\}, \quad h \geq 0.$$

Examples: Simulation



Realization of the sub-Gaussian random field

Examples: α–stable random fields

$S\alpha S$ Lévy motion

$$X(t) = \int_{[0,1]^d} 1\!\!1\{x_1 \le t_1, \dots, x_d \le t_d\} M(dx),$$

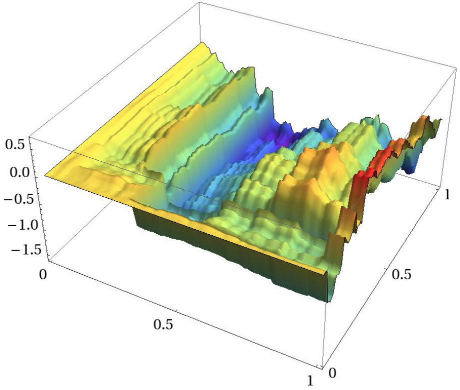
where $t = (t_1, \dots, t_d)^\top \in [0,1]^d$, and M is a $S\alpha S$ random measure with Lebesgue control measure.

Simulation: Two-dimensional $S\alpha S$ Lévy motion

$$X(t) = \int_{[0,1]^2} 1\!\!1\{x_1 \le t_1, x_2 \le t_2\} M(dx), \quad t \in [0,1]^2,$$

where $\alpha = 1.5$.

Examples: Simulation



Realization of the Lévy stable motion

Stable random fields: Properties and characteristics

Let

$$X(t) = \int_E f_t(x) M(dx), \; t \in \mathbb{R}^d$$

as above.

- Symmetry: if $\beta(x) = 0 \; \forall x$ then the field X is symmetric.
- Scale parameter of $X(t)$: $\sigma_{X(t)} = \|f_t\|_{L^\alpha}$ where

$$\left(\mathbb{E}|X(t)|^p\right)^{1/p} = c_{\alpha,\beta}(p) \cdot \sigma_{X(t)}$$

  for $0 < p < \alpha$, $0 < \alpha < 2$ and some constant $c_{\alpha,\beta}(p)$.
- Covariation function: for $t_1, t_2 \in \mathbb{R}^d$ and $1 < \alpha \leq 2$

$$\kappa(t_1, t_2) = [X(t_1), X(t_2)]_\alpha = \int_E f_{t_1}(x) f_{t_2}(x)^{<\alpha-1>} m(dx).$$

Stable random fields: Properties and characteristics

- Stationarity: if $E = \mathbb{R}^d$, $f_t(x) = f(t-x)$, $x, t \in \mathbb{R}^d$, $\beta(x) = const$ and $m(dx) = dx$ then X is stationary (moving average) and $\kappa(s,t) = \kappa(s-t, o) = \kappa(h)$, $h = s-t$, $s, t \in \mathbb{R}^d$.
- Linear dependence: For a d-dimensional α-stable random vector $\boldsymbol{X} = (X_1, \ldots, X_d)^\mathsf{T}$ with integral representation

$$\left(\int_E f_1(x) M(dx), \ldots, \int_E f_d(x) M(dx) \right)^\mathsf{T},$$

  let Γ be its spectral measure. $\boldsymbol{X}$ is not full-dimensional (i.e., Γ is concentrated on a great sub–sphere of S_d) iff $\sum_{i=1}^d c_i X_i = 0$ a.s. for some $(c_1, \ldots, c_d)^\mathsf{T} \in \mathbb{R}^d \setminus \{0\}$. This is equivalent to $\sum_{i=1}^d c_i f_i(x) = 0$ m-a. e.

Part 4: Prediction of stable random functions

- (Non)linear predictors and their properties
  - Least scale predictor
  - Covariation orthogonal predictor
  - Maximization of covariation
- Numerical results
- Open problems
- Literature

Prediction of stable random functions

Random functions without finite second moments:

- discrete stable processes: minimization of dispersion (Cambanis, Soltani (1984); Brockwell, Cline (1985); Kokoszka (1996); Brockwell, Mitchell (1998); Gallardo et al. (2000); Hill (2000))
- fractional stable motion: conditional simulation (Painter(1998))
- subgaussian random functions: maximum likelihood (ML) (Painter(1998)), linear regression (Miller (1978)), conditional simulation
- stable moving average processes: minimization of L^1-distance (Mohammadia, Mohammadpour (2009))
- α-stable random fields with integral spectral repr.: three methods (Karcher, Shmileva, S. (2013))

Prediction

- Let X be a centered ($\mathbb{E}\, X(t) = 0$, $t \in \mathbb{R}^d$) α-stable random field, $1 < \alpha \leq 2$, with skewness intensity β satisfying the spectral representation

$$X(t) = \int_E f_t(x) M(dx), \quad t \in \mathbb{R}^d.$$

- Let $X(t_1), \ldots, X(t_n)$ be the observations of X for $t_1, \ldots t_n \in W$, $W \subset \mathbb{R}^d$ being a compact set.
- Non-linear predictors for $X(t)$, $t \notin \{t_1, \ldots, t_n\}$: for some particular random functions (e.g. subgaussian ones) one can use
  - Maximum likelihood (ML) predictors
  - Conditional simulators

Linear predictors

- Linear predictor for $X(t)$, $t \notin \{t_1, \ldots, t_n\}$:

$$\widehat{X}(t) = \sum_{i=1}^{n} \lambda_i X(t_i),$$

where $\lambda_i = \lambda_i(t, t_1, \ldots, t_n)$ for $i = 1, \ldots, n$.

Properties

- $\widehat{X}$ is unbiased since $\mathbb{E}\,\widehat{X}(t) = 0$, $t \in \mathbb{R}^d$
- $\widehat{X}$ is exact if $\widehat{X}(t_i) = X(t_i)$ a.s., $i = 1, \ldots, n$.
- $\widehat{X}$ is continuous if $\lambda_i = \lambda_i(\cdot, t_1, \ldots, t_n)$ are continuous as functions of t, $i = 1, \ldots, n$

Linear predictors

$\widehat{X}(t)$ should be optimal in a sense that it

- minimizes the scale parameter $\sigma_{\widehat{X(t)}-X(t)}$
  $\Longrightarrow$ Least Scale Linear (LSL) Predictor
- mimics the covariation structure between $X(t)$ and $X(t_j)$, $j = 1, \ldots, n$
  $\Longrightarrow$ Covariation Orthogonal Linear (COL) Predictor
- maximizes the covariation between $X(t)$ and $\widehat{X}(t)$
  $\Longrightarrow$ Maximization of Covariation Linear (MCL) Predictor

Least Scale Linear Predictor

Generalization of Kriging techniques:

$$\sigma^{\alpha}_{\widehat{X(t)}-X(t)} = \int_E \left| f_t(x) - \sum_{i=1}^{n} \lambda_i f_{t_i}(x) \right|^{\alpha} m(dx) \rightarrow \min$$

with respect to $\lambda_1, \ldots, \lambda_n$.

Non-linear optimization problem $\Longrightarrow$ numerical methods for its solution.

Least Scale Linear Predictor

Lemma

Let $\alpha \in (1, 2)$. A solution of the above minimization problem resolves the system of equations

$$\left[X(t_j), X(t) - \sum_{i=1}^{n} \lambda_i X(t_i) \right]_\alpha = 0, \quad j = 1, \ldots n,$$

which can be written as

$$\int_E f_{t_j}(x) \left(f_t(x) - \sum_{i=1}^{n} \lambda_i f_{t_i}(x) \right)^{\langle \alpha - 1 \rangle} m(dx) = 0, \quad j = 1, \ldots n.$$

This is a system of non–linear equations in $\lambda_1, \ldots, \lambda_n$.

Least Scale Linear Predictor

Theorem

- *Existence: The LSL estimator exists.*
- *Uniqueness: Assume that the random vector $(X(t_1), \ldots, X(t_n))^\top$ is full-dimensional. Then the LSL estimator is unique.*
- *Exactness: If there is a unique LSL estimator, then it is obviously exact.*
- *Continuity: If the random field X is stochastically continuous and $(X(t_1), \ldots, X(t_n))^\top$ is full-dimensional then the LSL estimator is continuous.*

Least Scale Linear Predictor

Example: $S\alpha S$ Lévy motion

$X(t) = \int_0^\infty 1\!\!1(x \le t) M(dx)$, where M is a $S\alpha S$ random measure with Lebesgue control measure. Let $t = 3/4$ and $t_1 = 1$. Then the optimization problem for the LSL predictor is

$$\begin{aligned} \sigma^\alpha_{\widehat{X(t)}-X(t)} &= \int_0^{3/4} |1-\lambda_1|^\alpha dx + \int_{3/4}^1 |\lambda_1|^\alpha dx \\ &= \frac{3}{4}|1-\lambda_1|^\alpha + \frac{1}{4}|\lambda_1|^\alpha \to \min_{\lambda_1}. \end{aligned}$$

We obtain the LSL predictor

$$\widehat{X(t)} = \frac{1}{1+(1/3)^{1/(\alpha-1)}} X(t_1).$$

Covariation Orthogonal Linear Predictor

Let X be a random field as above. The linear predictor with weights $\lambda_1, \ldots, \lambda_n$ being a solution of the following system of equations

$$[X(t), X(t_j)]_\alpha = [\widehat{X(t)}, X(t_j)]_\alpha, \quad j = 1, \ldots, n$$

is the COL predictor. It is a linear system of equations

$$\left[X(t), X(t_j)\right]_\alpha - \sum_{i=1}^{n} \lambda_i \left[X(t_i), X(t_j)\right]_\alpha = 0, \quad j = 1, \ldots, n.$$

The COL predictor is obviously exact.

Covariation Orthogonal Linear Predictor

The regression of $X(t)$ on $(X(t_1), \dots, X(t_n))^\mathsf{T}$ is called linear if there exists some $(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ such that it holds a.s.

$$\mathbb{E}(X(t)|X(t_1), \dots, X(t_n)) = \sum_{i=1}^{n} \lambda_i X(t_i).$$

The regression of $X(t)$ on $(X(t_1), \dots, X(t_n))^\mathsf{T}$ is linear if X is e.g. a (sub)Gaussian random function.

Lemma

If the regression of $X(t)$ on the random vector $(X(t_1), \dots, X(t_n))^\mathsf{T}$ is linear then the vector $(\lambda_1, \dots, \lambda_n)^\mathsf{T}$ is a solution of the COL system of equations.

Covariation Orthogonal Linear Predictor

Theorem

Let X be an α-stable moving average.

- *If the kernel function $f : \mathbb{R}^d \to \mathbb{R}_+$ is positive semi-definite, then the covariation function κ is positive semi-definite. If $f : \mathbb{R}^d \to \mathbb{R}_+$ is positive definite and positive on a set with positive Lebesgue measure, then κ is positive definite.*
- *If the covariation function is positive definite then the COL predictor exists and is unique.*
- *If the covariation function is positive definite and continuous, then the COL predictor is continuous.*

Covariation Orthogonal Linear Predictor

Proof.

The weights of the COL predictor satisfy the system of equations

$$\begin{pmatrix} \kappa(0) & \cdots & \kappa(t_n - t_1) \\ \vdots & \ddots & \vdots \\ \kappa(t_n - t_1) & \cdots & \kappa(0) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} \kappa(t - t_1) \\ \vdots \\ \kappa(t - t_n) \end{pmatrix}$$

□

Covariation Orthogonal Linear Predictor

Example: $S\alpha S$ Ornstein-Uhlenbeck process.

$$X(t) = \int_{\mathbb{R}} e^{-\lambda(t-x)} \mathbb{1}(t - x \geq 0) M(dx), \quad t \in \mathbb{R},$$

for some $\lambda > 0$, where M is a $S\alpha S$ random measure with Lebesgue control measure. If $t_1 < t_2 < \ldots < t_n < t$, then the regression of $X(t)$ on $(X(t_1), \ldots, X(t_n))^{\mathsf{T}}$ is linear, and $\widehat{X(t)} = e^{-\lambda(t-t_n)} X(t_n)$.

Covariation Orthogonal Linear Predictor

Let X be a centered (sub)Gaussian α-stable random field with covariance function C of the Gaussian part.
Then

$$[X(t_i), X(t_j)]_\alpha = 2^{-\alpha/2} C(t_i - t_j) C(0)^{(\alpha-2)/2}.$$

The COL predictor is the solution of the system

$$\begin{pmatrix} C(0) & \cdots & C(t_n - t_1) \\ \vdots & \ddots & \vdots \\ C(t_n - t_1) & \cdots & C(0) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} C(t - t_1) \\ \vdots \\ C(t - t_n) \end{pmatrix}$$

and thus coincides with simple kriging.

Covariation Orthogonal Linear Predictor

Theorem

Let X be a centered (sub)Gaussian α-stable random field with positive definite covariance function C of the Gaussian part.

- *The COL predictor exists and is unique.*
- *If the covariance function is continuous, then the COL predictor is continuous.*

Theorem

For (sub)Gaussian random fields, the COL and LSL predictors for $X(t)$ coincide (with the maximum–likelihood (ML) estimator of $X(t)$).

Maximization of Covariation Linear Predictor

Let X be an α-stable random field with spectral integral representation and $\alpha > 1$. To construct the MCL predictor, solve

$$\begin{cases} \left[\widehat{X(t)}, X(t)\right]_\alpha = \sum_{i=1}^n \lambda_i \left[X(t_i), X(t)\right]_\alpha \to \max\limits_{\lambda_1,\dots,\lambda_n}, \\ \sigma_{\widehat{X(t)}} = \sigma_{X(t)}, \end{cases}$$

where the condition $\sigma_{\widehat{X(t)}} = \sigma_{X(t)}$ means $\widehat{X(t)} \stackrel{d}{=} X(t)$ for $S\alpha S$ random fields.

Maximization of Covariation Linear Predictor

Theorem

Assume that the random vector $(X(t_1), \ldots, X(t_n))^{\mathsf{T}}$ is full-dimensional.

- *Existence: The MCL predictor exists.*
- *Uniqueness: If $[X(t_i), X(t)]_\alpha \neq 0$ for some $i \in \{1, \ldots, n\}$ then the MCL predictor is unique.*
- *Exactness: If the MCL predictor is unique then it is exact.*
- *Continuity: If X is a moving average, the covariation function κ is continuous and $\kappa(t_i - t) \neq 0$ for some $i \in \{1, \ldots, n\}$ then the MCL predictor is continuous.*

Numerical results

Two-dimensional $S\alpha S$ Lévy motion

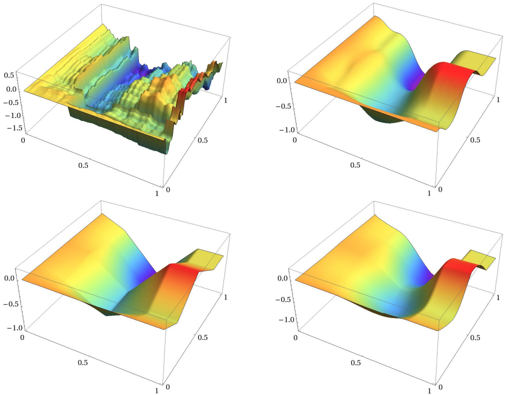
$$X(t) = \int_{[0,1]^2} \mathbb{1}\{x_1 \leq t_1, x_2 \leq t_2\} M(dx), \quad t \in [0,1]^2,$$

where M is a $S\alpha S$ random measure with $m =$ Lebesgue control measure and $\alpha = 1.5$.

Method	5%-Quantile	1st Quartile	Median	3rd Quartile	95%-Quantile
LSL	-0.5170	-0.1246	0.0000	0.1226	0.5045
COL	-0.5263	-0.1289	0.0002	0.1266	0.5137
MCL	-0.6093	-0.1455	-0.0007	0.1407	0.5895

Summary statistics for the deviations $X(t) - \widehat{X(t)}$.

Numerical results



Realization of the Lévy stable motion (top left) and the extrapolations (out of 9 observation points) based on the LSL method (top right), the COL method (bottom left) and the MCL method (bottom right).

Numerical results

Subgaussian random field

$$X = \{A^{1/2}G(t), t \in [0,1]^2\}$$

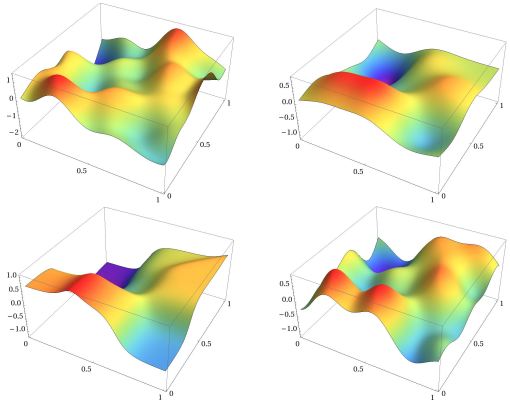
with $\alpha = 1.5$, $A \sim S_{\alpha/2}((\cos(\pi\alpha/4))^{2/\alpha}, 1, 0)$ and G being a stationary isotropic Gaussian random field with covariance function

$$C(h) = 7\exp\{-(h/0.1)^2\}, \quad h \geq 0.$$

Method	5%-Quantile	1st Quartile	Median	3rd Quartile	95%-Quantile
LSL (COL, ML)	-1.5451	-0.4446	0.0018	0.4503	1.5363
MCL	-1.8204	-0.4899	0.0046	0.5016	1.7580
CS	-2.7523	-0.5837	0.0058	0.5985	2.7262

Summary statistics for the deviations $X(t) - \widehat{X(t)}$.

Numerical results



Realization of the sub-Gaussian random field (top left) and the extrapolations (out of 9 observation points) based on the LSL (COL, ML) method (top right), the MCL method (bottom left) and the CS method (bottom right).

Open problems

- Extrapolation methods and their properties for stable random fields with $\alpha \in (0, 1]$
- Control of skewness of known predictors for non–symmetric stable random fields ($\beta \neq 0$)
- Characterization of the covariation function

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