Alexey A. Tuzhilin

Geometrical optimization problems with one-dimensional branching extremals

(in collaboration with Alexander O. Ivanov)

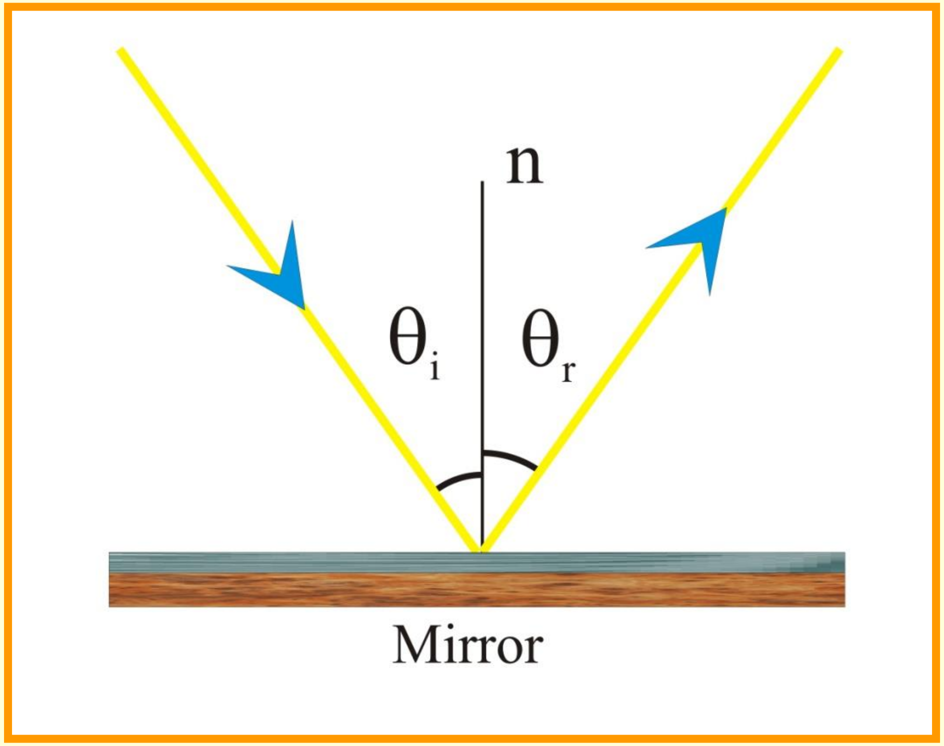
Pierre-Louis Moreau de Maupertuis
(1698-1759)

"If there occurs some change in nature, the amount of ***action*** *necessary for this change must be as small as possible"*

Pierre de Fermat (1601–1665)

The principle of least time (1662): *The path taken between two points by a ray of light is the path that can be traversed in the least time*.
As a consequence, one can deduce the reflection and refraction laws.

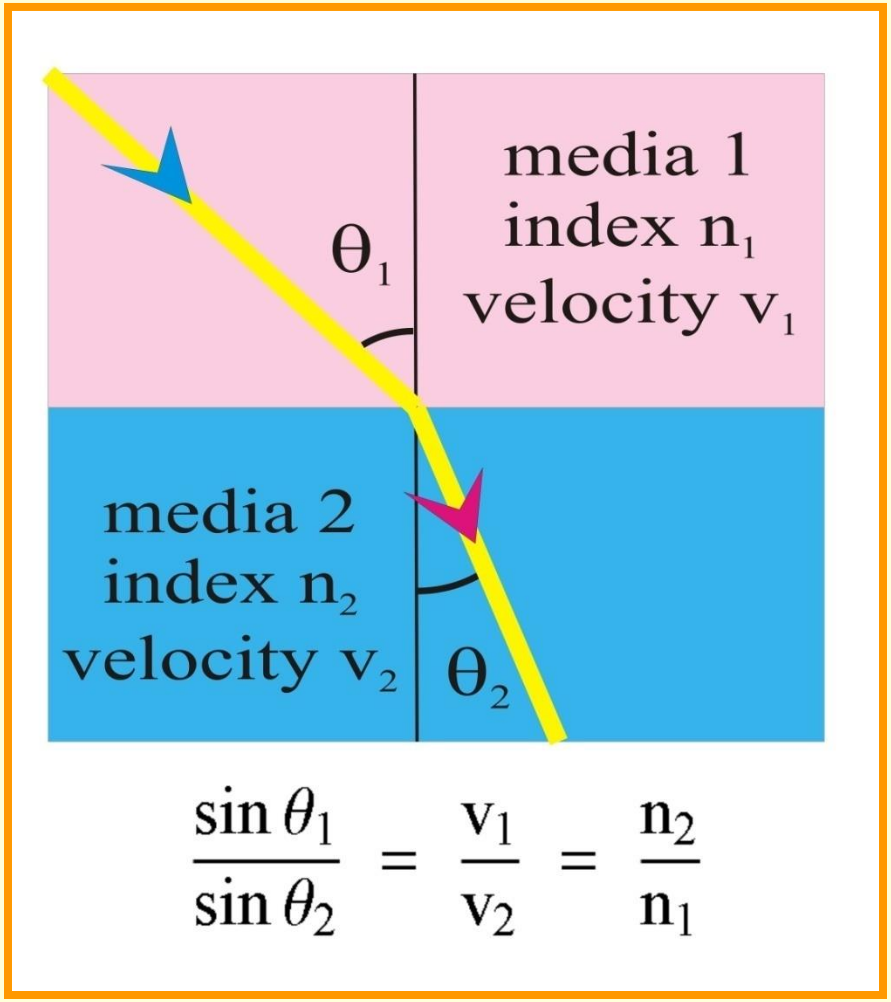
Reflection Law



$\theta_i = \theta_r$.

The angle of incidence equals the angle of reflection

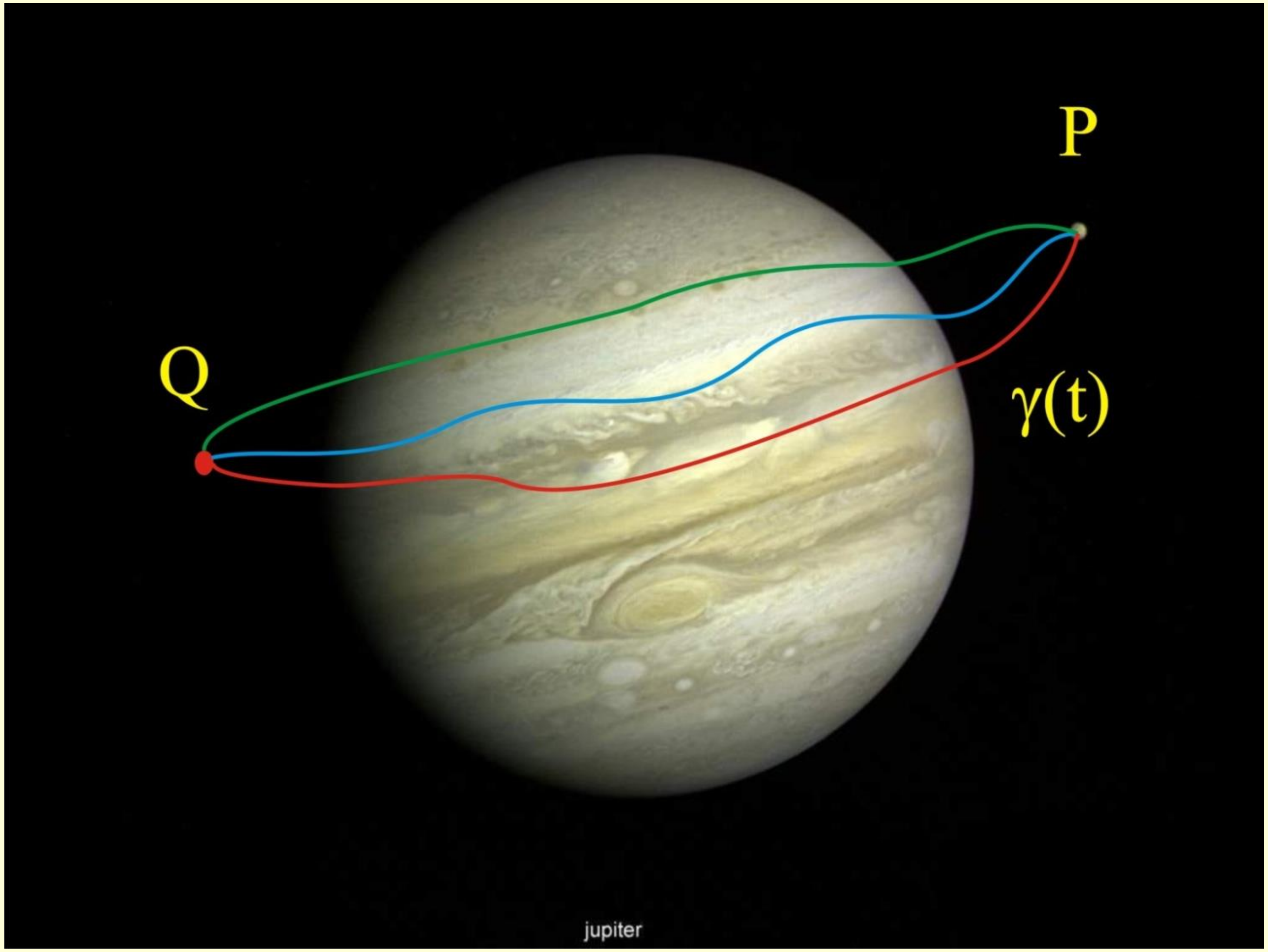
Willebrord Snel
van Royen (1580-1626)



The ratio of the sines of the angles of incidence and of refraction is a constant that depends on the media.

Leonhard Euler
(1707-1783)

Trajectories of point-mass motion in potential field of forces must minimize the integral of the difference between kinetic and potential energies



$U(x,y,z)$ is potential energy, $T = \frac{mv^2}{2} = \frac{m(\dot{x}^2+\dot{y}^2+\dot{z}^2)}{2}$ is kinetic energy

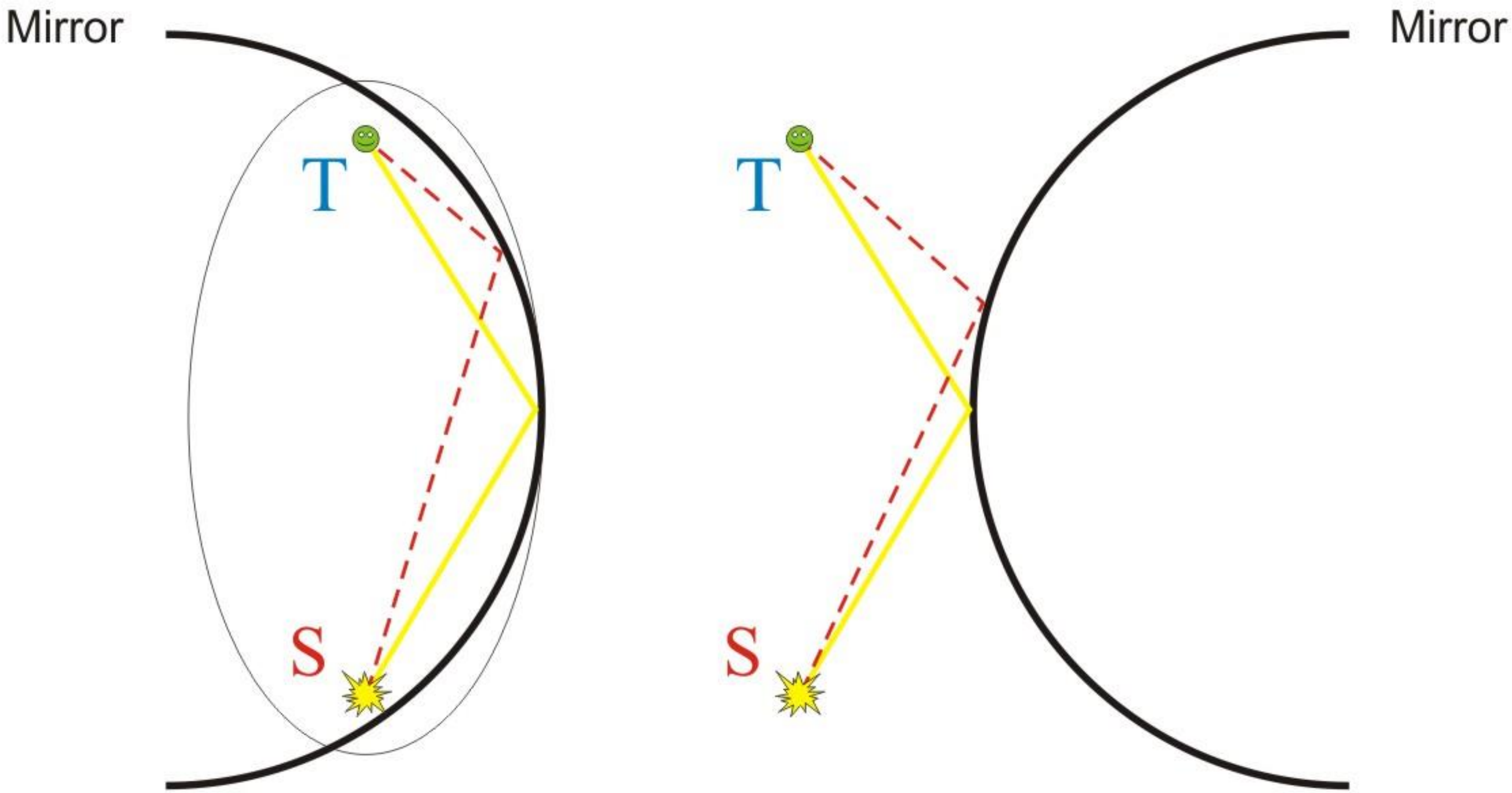
$L(x,y,z,\dot{x},\dot{y},\dot{z}) = T(\dot{x},\dot{y},\dot{z}) - U(x,y,z)$ is Lagrangian

$\gamma(t) = \big(x(t), y(t), z(t)\big)$, $a \le t \le b$, is a curve joining P and Q

The real trajectory minimizes the value

$$\Phi(\gamma) = \int_a^b L\big(x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)\big)\, dt$$

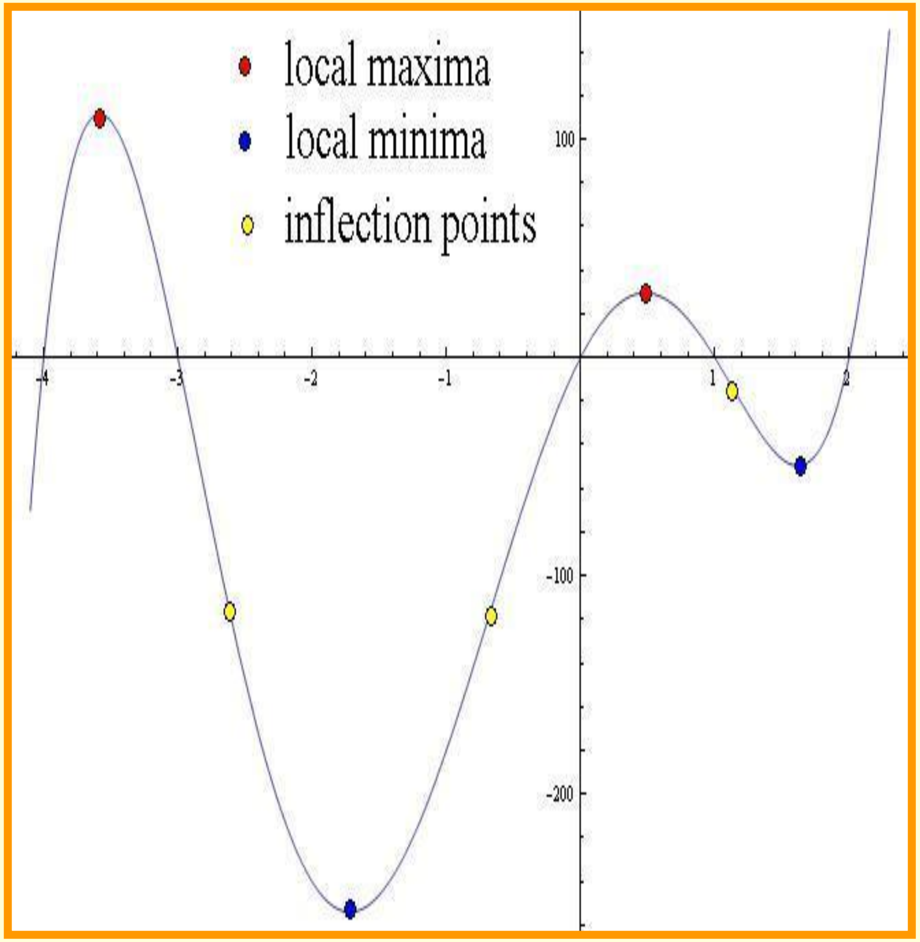
Ideas of Chevalier d'Arcy (1749)

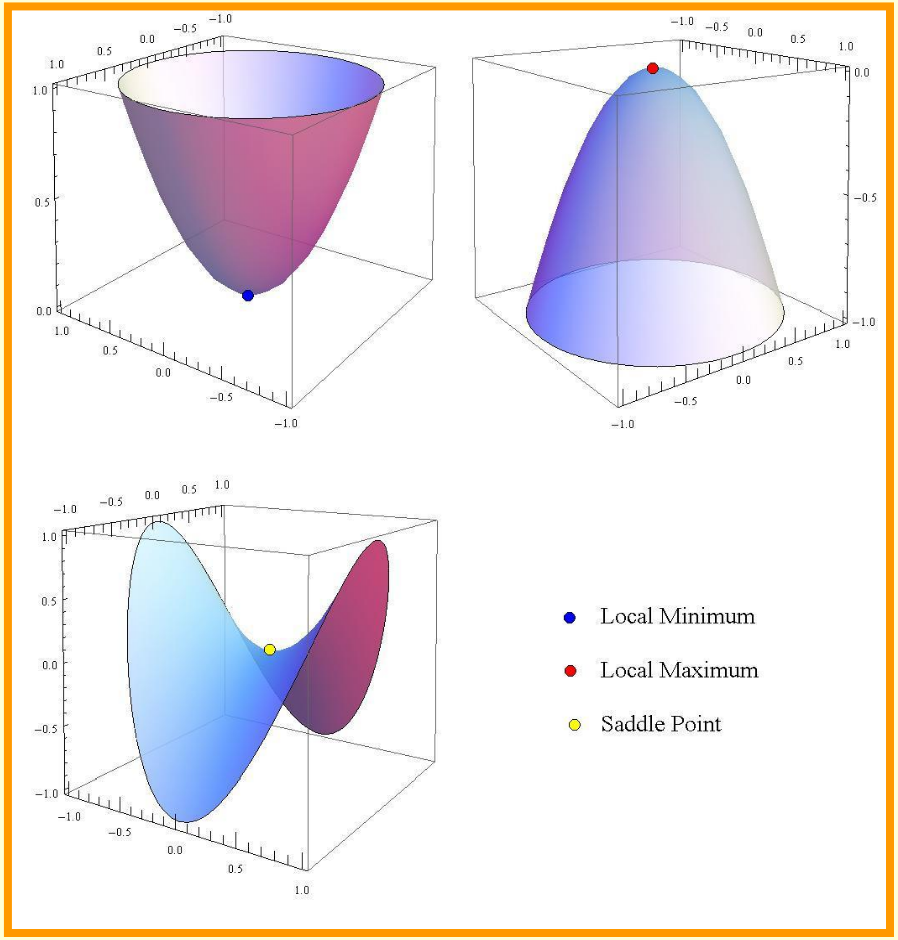


The yellow path is longer than the red one.
Here Nature is wasteful.

The yellow path is shorter than the red one.
Here Nature is parsimonious.

Various types of critical points



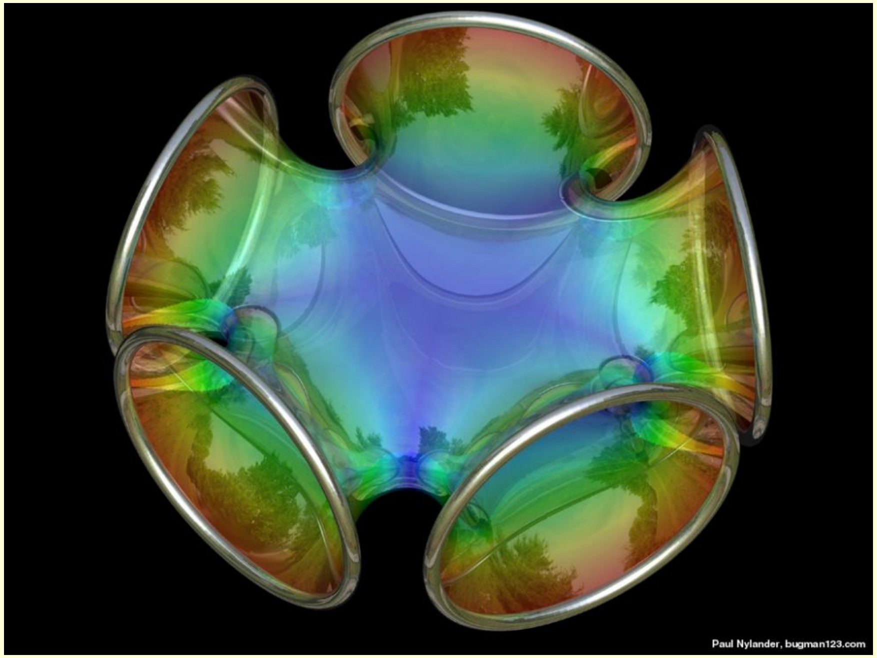


Soap films as extremals of the area functional (minimal surfaces).

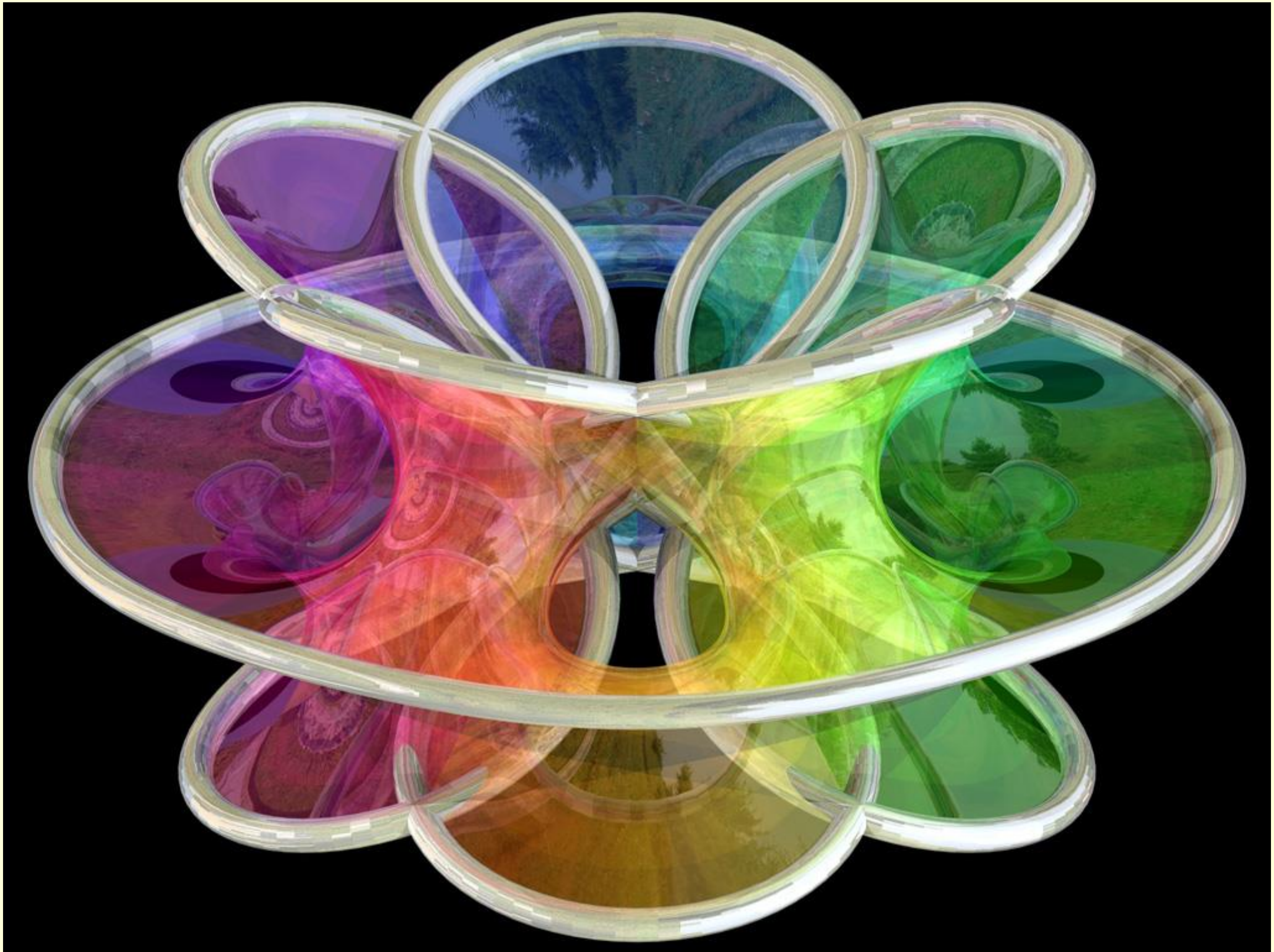
A soap film minimizes the surface tension which is proportional to the area of the film, thus, they minimize the area. Standard soap films correspond to local minima of the area functional because they are stable.

Costa minimal surface.

Jorge-Meeks k-noid.

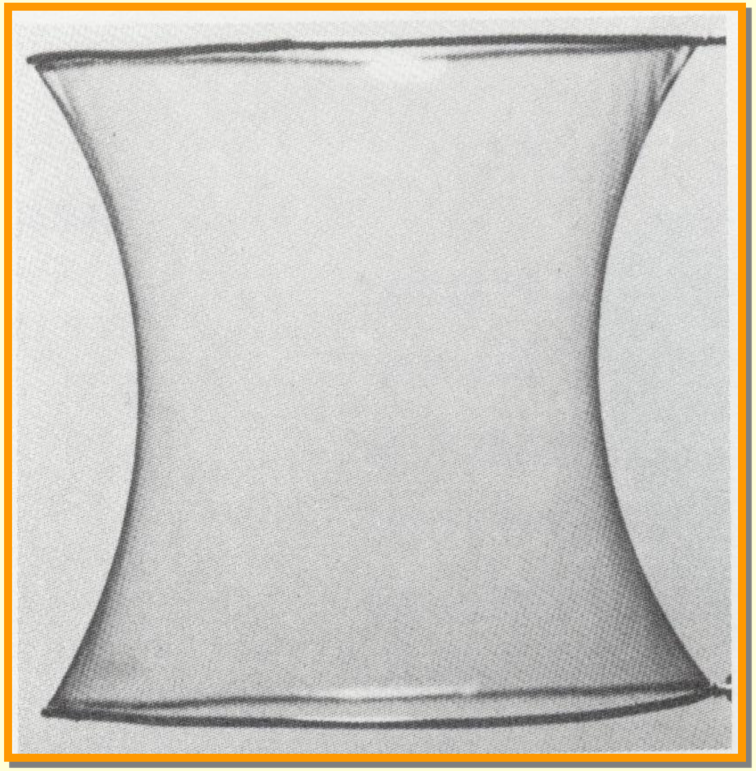


Richmond surface

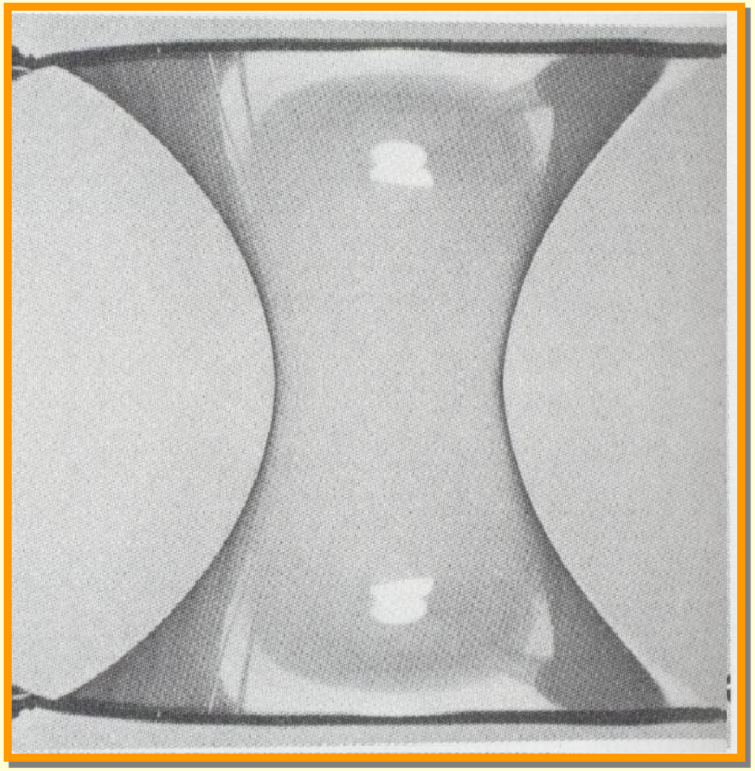


The “saddle” critical points of the functional correspond to unstable soap films which rarely to observe and hard to obtain.

Catenoids are the only minimal surfaces of revolution.



Stable catenoid



Unstable catenoid

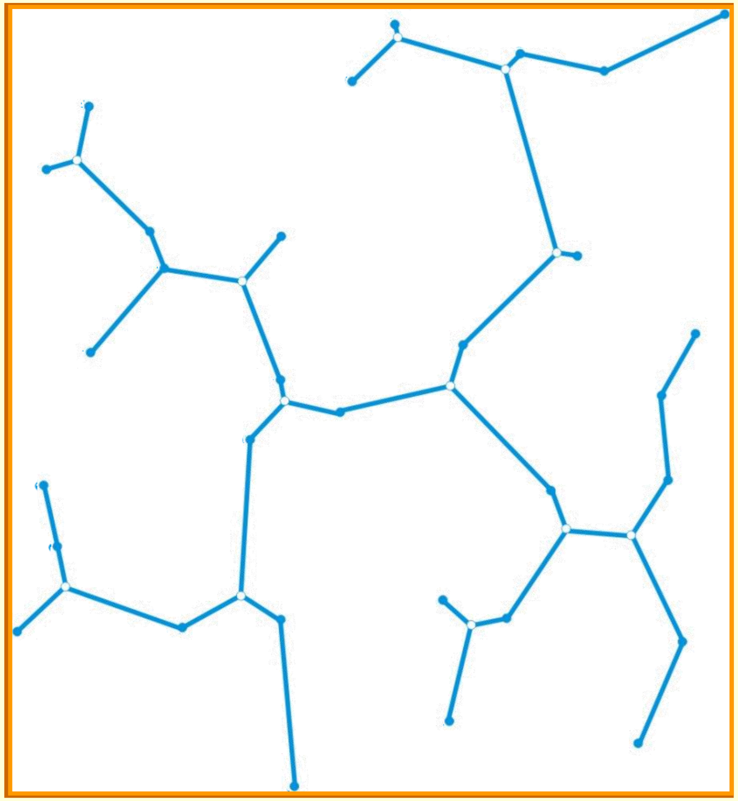
What are the methods to investigate such systems?

We'll show a few of them on the example of the famous Steiner Problem.

Steiner Problem

Construct a shortest network joining a given finite subset of the plane called the boundary.

Indeed, this problem was stated by Jarnik and Kössler in 1934



Shortest networks are called Steiner minimal trees or shortest trees

Transportation Problem and Steiner Problem

Steiner Problem on Manhattan plane (Rectilinear Steiner Problem) and chip design

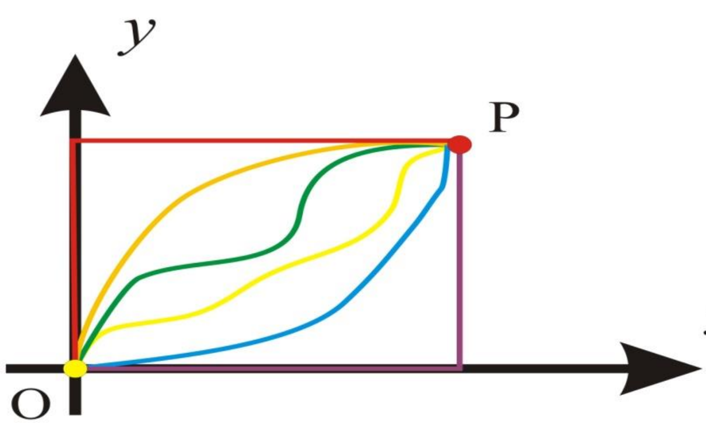
For $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$

Euclidean distans is

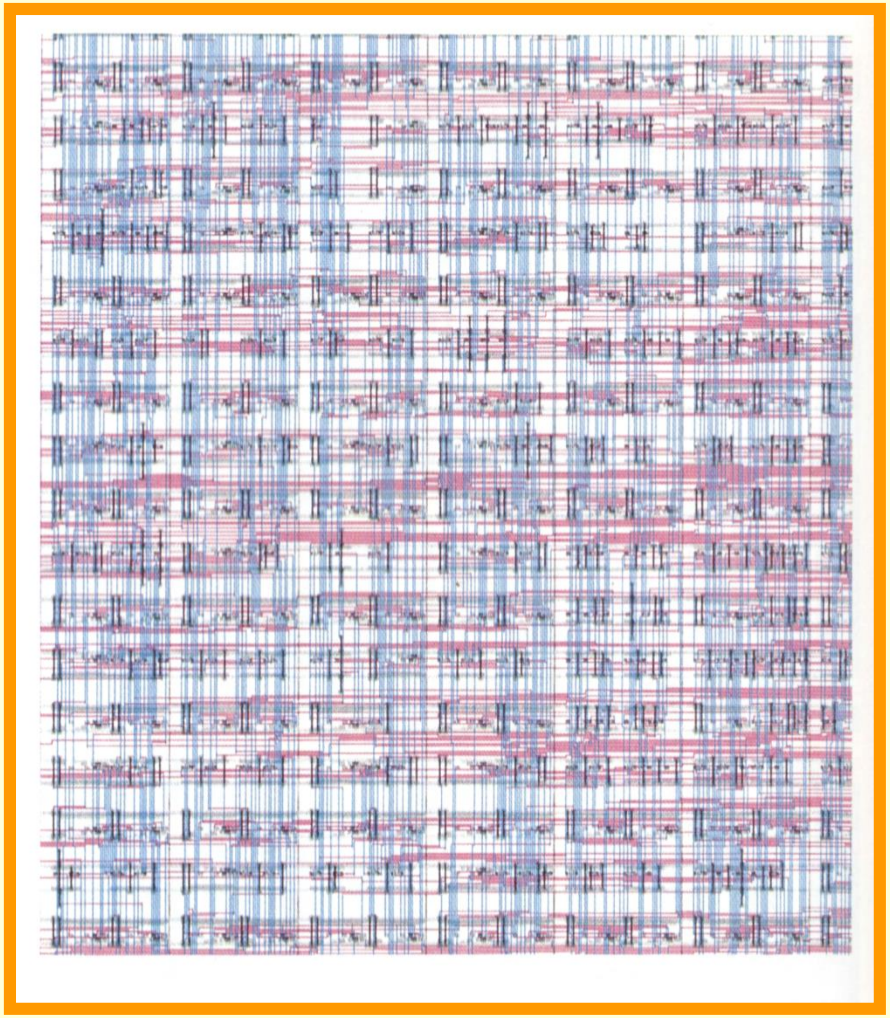
$$\rho_2(A_1, A_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Manhattan distance is

$$\rho_1(A_1, A_2) = |x_1 - x_2| + |y_1 - y_2|$$



All monotonic curves joining O and P have the same Manhattan distance



Steiner Problem in the space of words (phylogenetic tree)

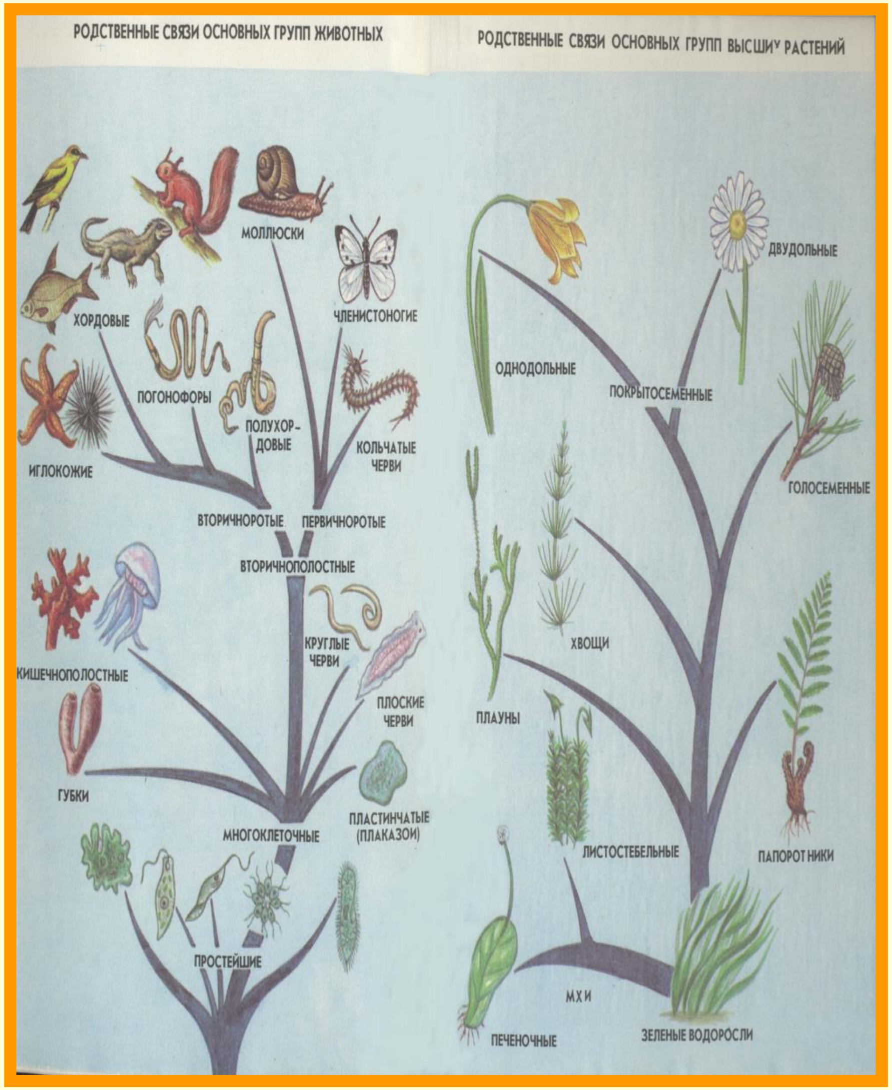
Elementary editor operations on a word are

insertion	**deletion**	**substitution**
$abcd \to ab\mathrm{x}cd$	$ab\mathrm{x}d \to abd$	$ab\mathrm{x}d \to ab\mathrm{y}d$

Hamming distance between two words w_1 and w_2 is the least number of elementary editor operations to pass from w_1 to w_2

To measure the difference between two species one can code them by words, for example, 4-letter DNA word, or 20-letter protein word, or a word characterizing the presence of different phenotypic properties, etc., and to calculate the Hemming distance between these words.

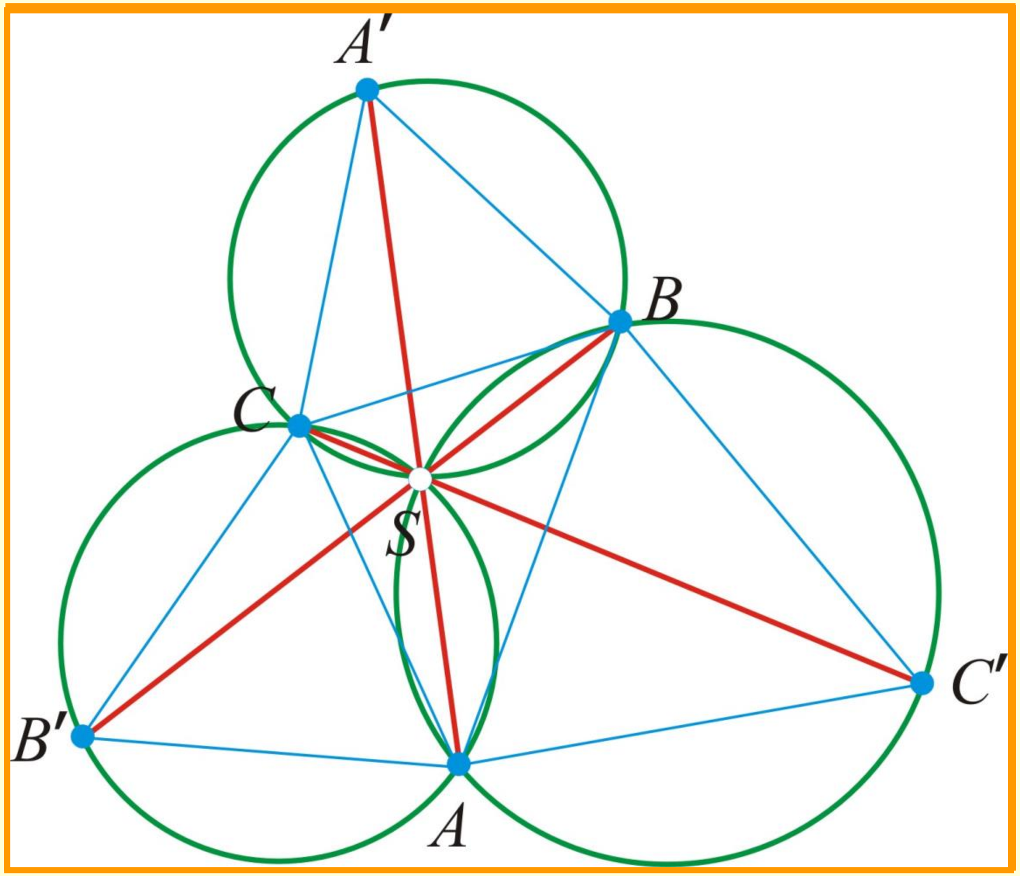
Biological assumption: evolution was optimal in the sense of minimization of the changes number (for example, minimization of mutations number). Thus, the evolution tree has to be the shortest tree (in Hemming distance) joining the words corresponding to noway day species. Thus enables to reconstruct the properties of predecessors.



Fermat Problem

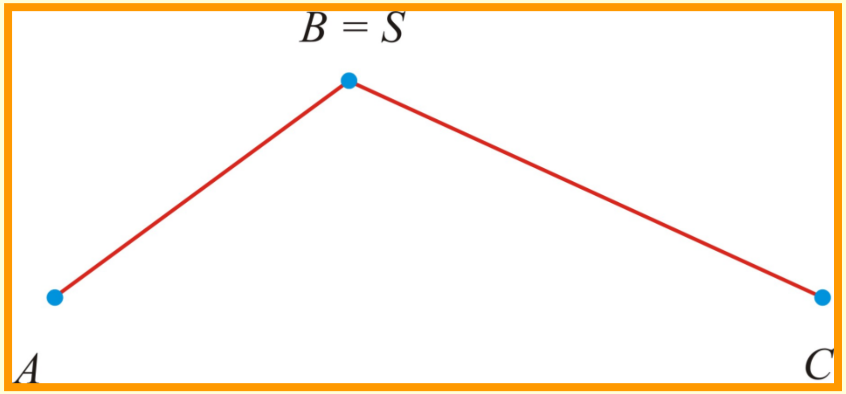
Given three points A, B, and C in the plane, find a point S such that the total distance from S to A, B, and C is minimal.

All the angles of ABC are less than 120°



General solution

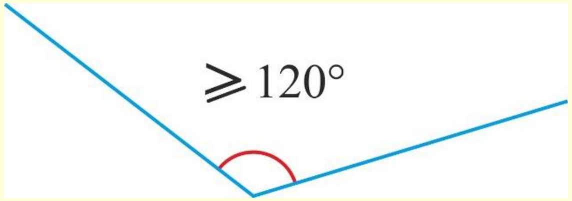
The angle B of ABC is at least 120°



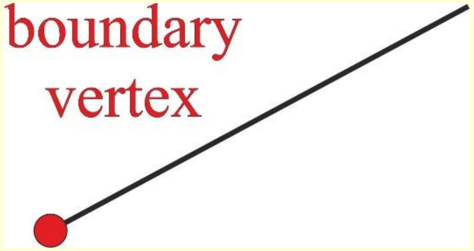
Local Structure of Shortest trees

Theorem (on the local structure of shortest trees in the plane).

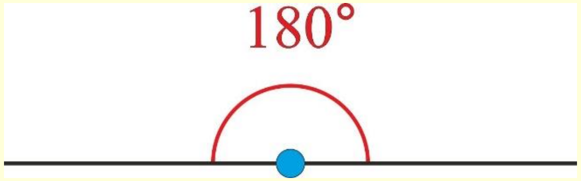
(1) Each shortest tree consists of straight segments meeting by the angles of at least 120°. In particular, the degree of any vertex of such a tree does not exceed 3.



(2) All degree one vertices belong to the boundary.



(3) If a vertex of degree two does not belong to the boundary, then the angle between two edges incident to it equals 180°.



Remarks.

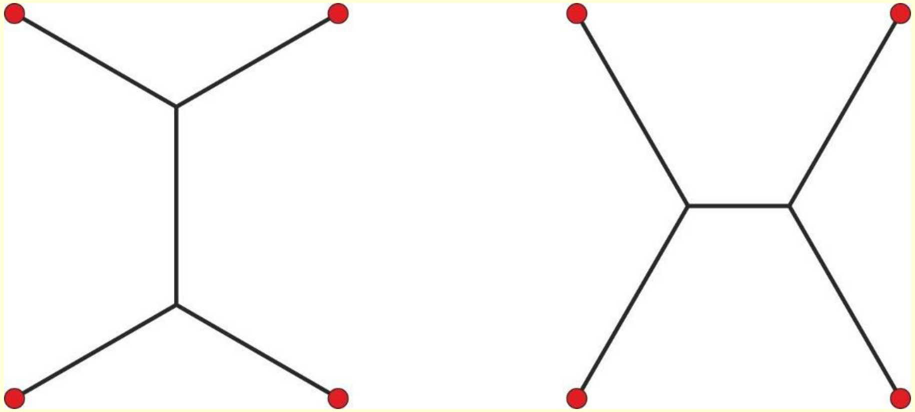
Vertices which do not belong to the boundary are called Steiner points or movable vertices.

Movable vertices of degree 2 can be as removed from, so as added to a shortest tree, without violating the minimality property of the tree, and one usually assumes that a shortest tree does not contain them.

If a planar graph (not necessary a tree) possesses all the properties from the theorem on local structure of Steiner minimal trees, then it is called a local minimal netwrok.

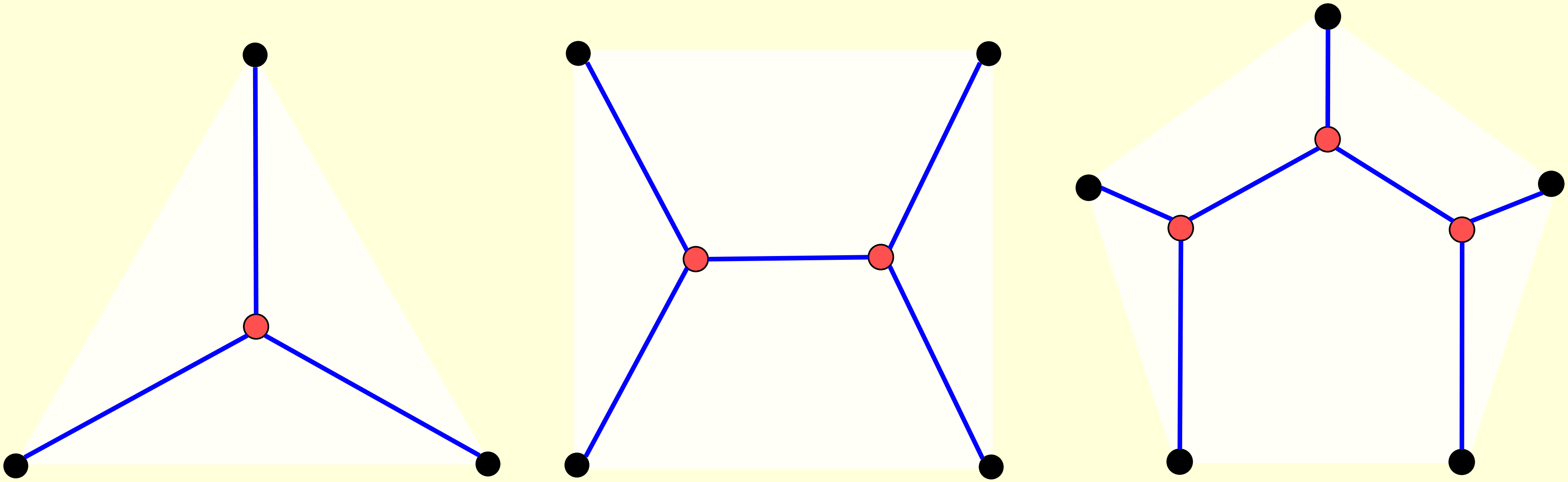
Each shortest tree is local minimal. The converse is not true.

This is shortest



This is not shortest

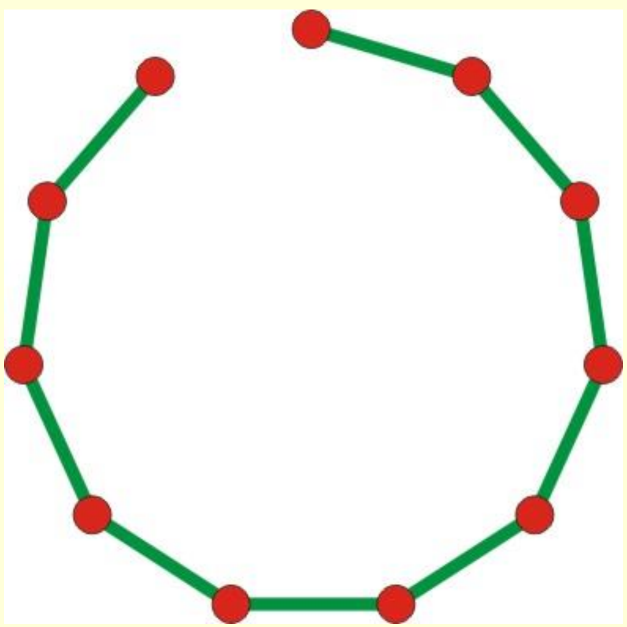
Shortest trees joining the vertices of regular n-gons for n = 3, 4, 5.

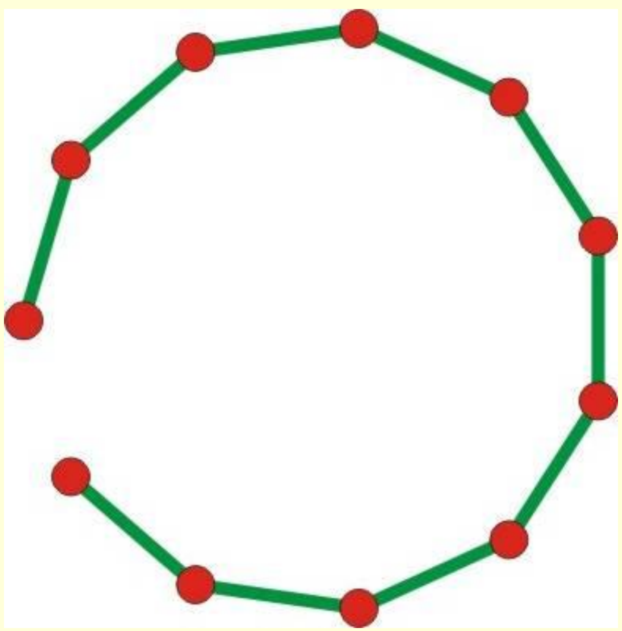


n = 3 n = 4 n = 5

Shortest trees joining the vertices of regular n-gons for n ≥ 6

Theorem (V.Jarnik, O.Kössler, D.Z.Du, F.K.Hwang, J.F.Weng). Given $n \geq 6$, each shortest tree joining vertices of a regular n-gon consists of all sides of the n-gon, except any one.

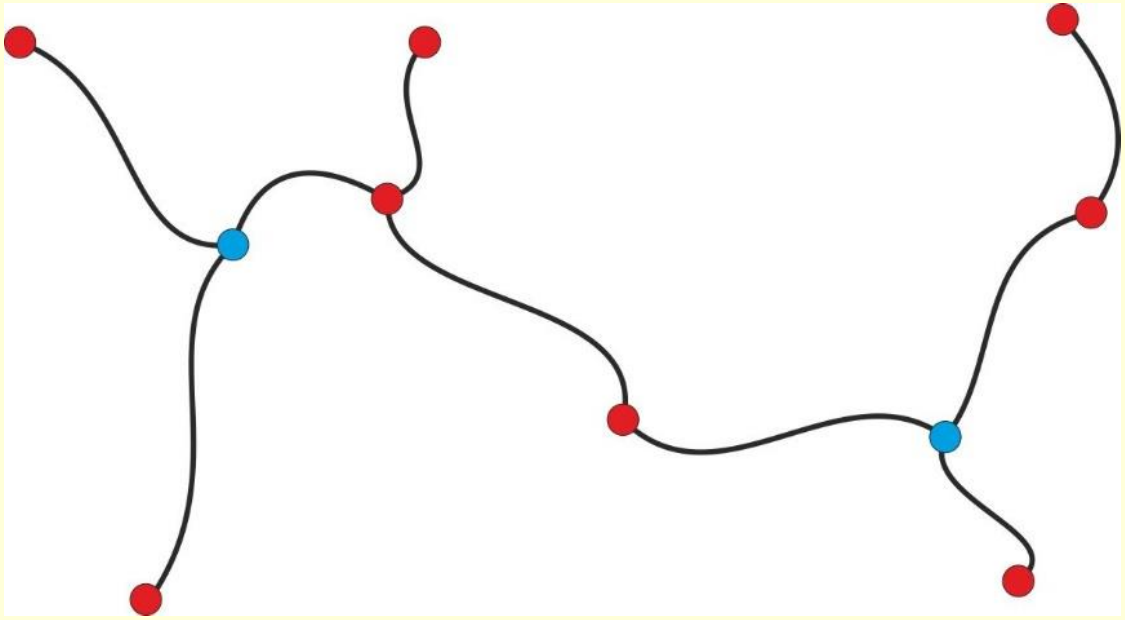




How to consruct all shortest trees joining a given points set M?

Choose from the set of all local minimal trees joining M the shortest ones.

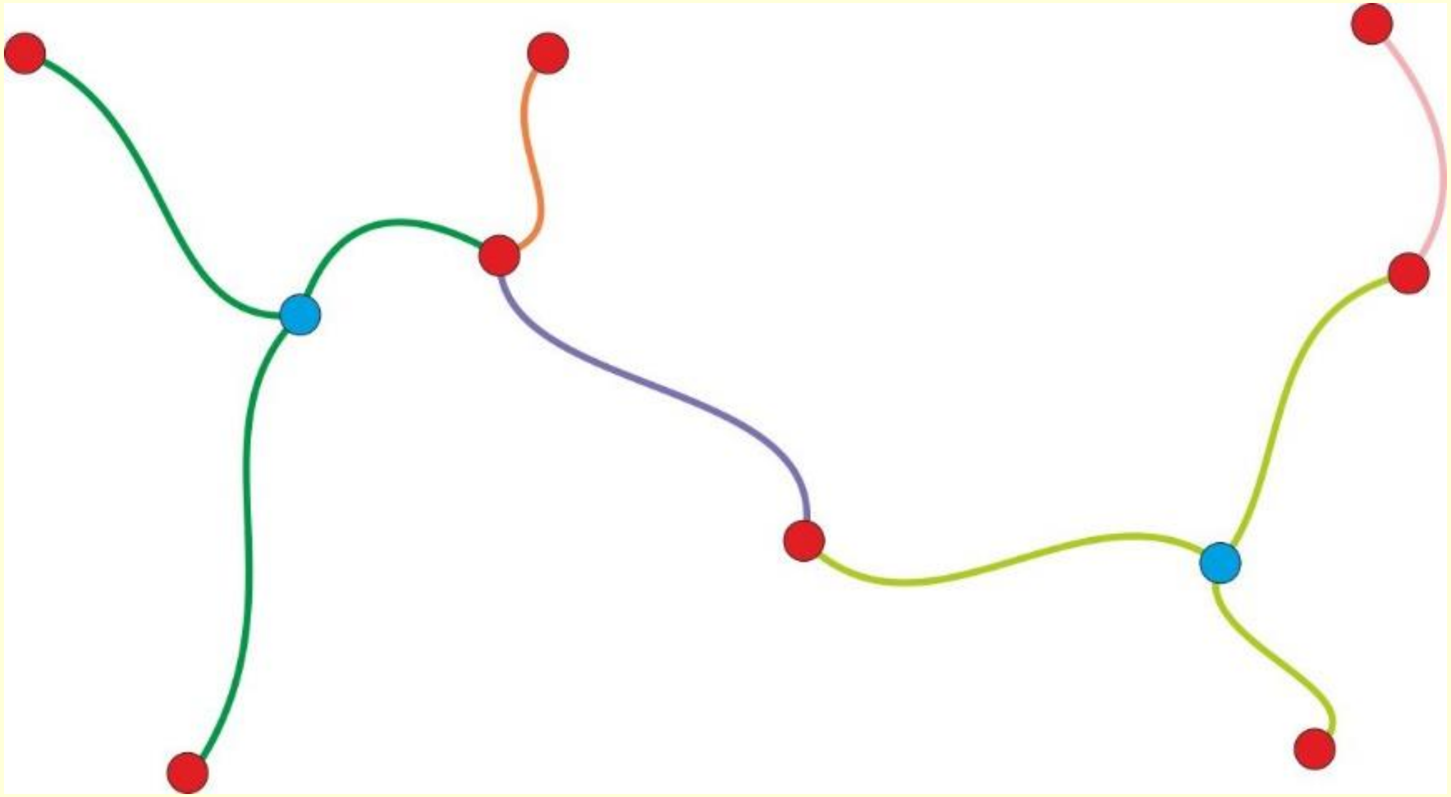
Possible structures of local minimal trees : Steiner trees, namely,
vertices degrees ≤ 3;
all vertices with degree 1 and 2 belong to M.



Cut the tree at boundary vertices of degree more than 1
we decompose it into binary components

Each binary component:

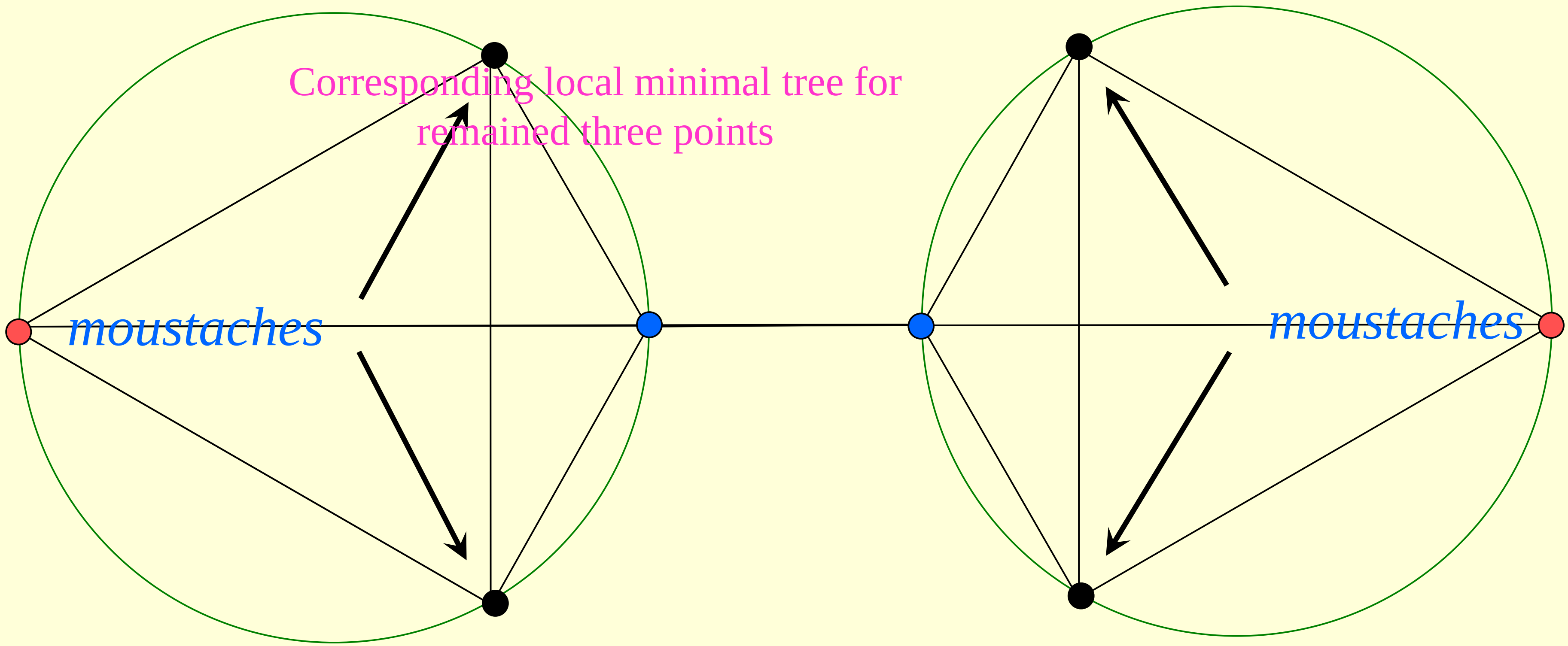
- does not have degree 2 vertices
- its boundary is just all the vertices of degree 1



5 binary components

Melzak algorithm (1960)

It constructs (if possible) the local minimal binary tree of a given structure?

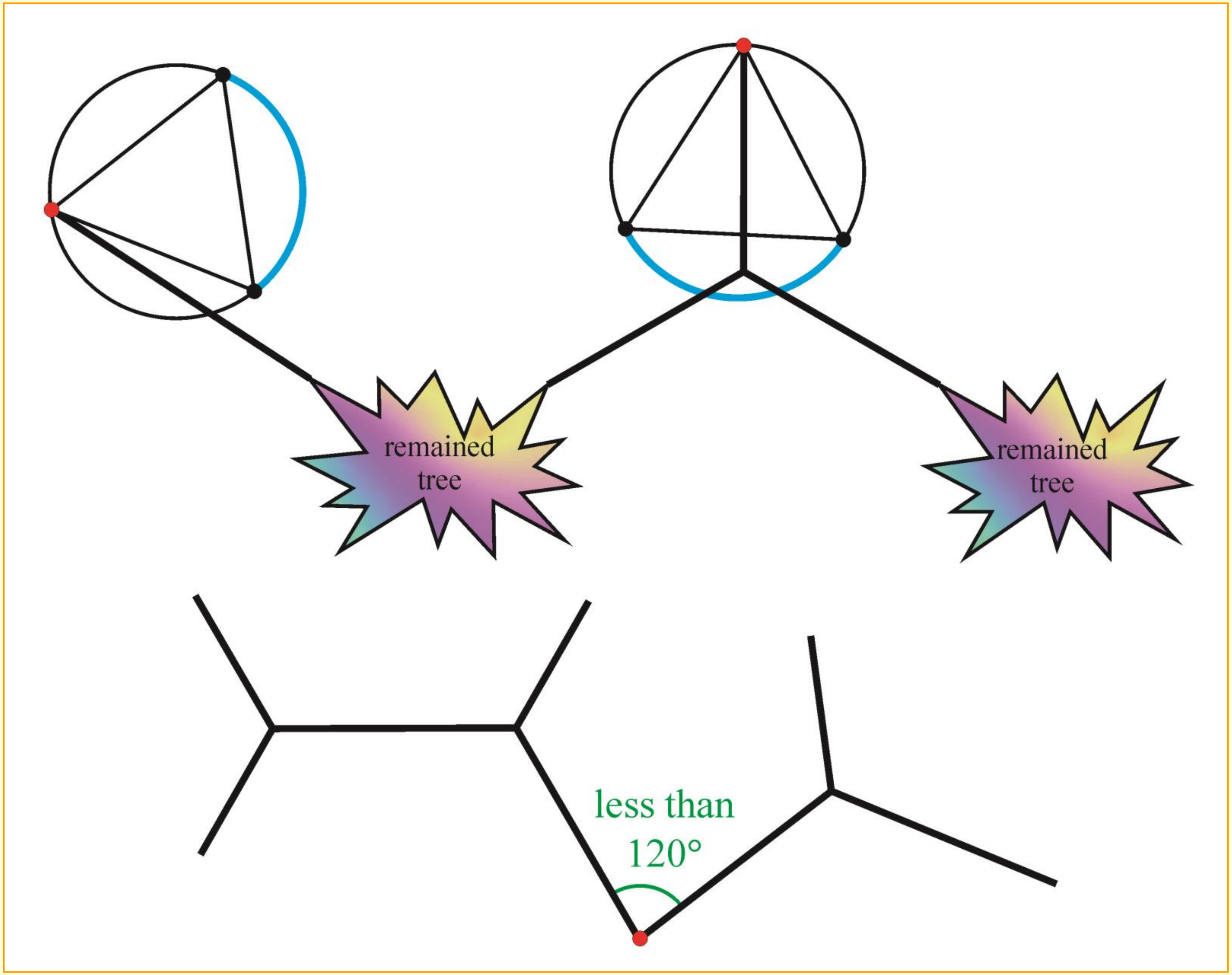


The reverse trace of Melzak algorithm

The forward trace of Melzak algorithm

The network has been reconstructed.

Three main obstacles to construct a local minimal tree



Steiner problem is NP-hard

For a subset M of the plane consisting of n points the number of all different (non-equivalent) plane binary trees joining M equals $\frac{(2n-4)!}{2^{n-2}(n-2)!}$.

Thus, the complete list of the combinatorial structures pretending to be the ones for shortest trees on M grows very fast as n increases.

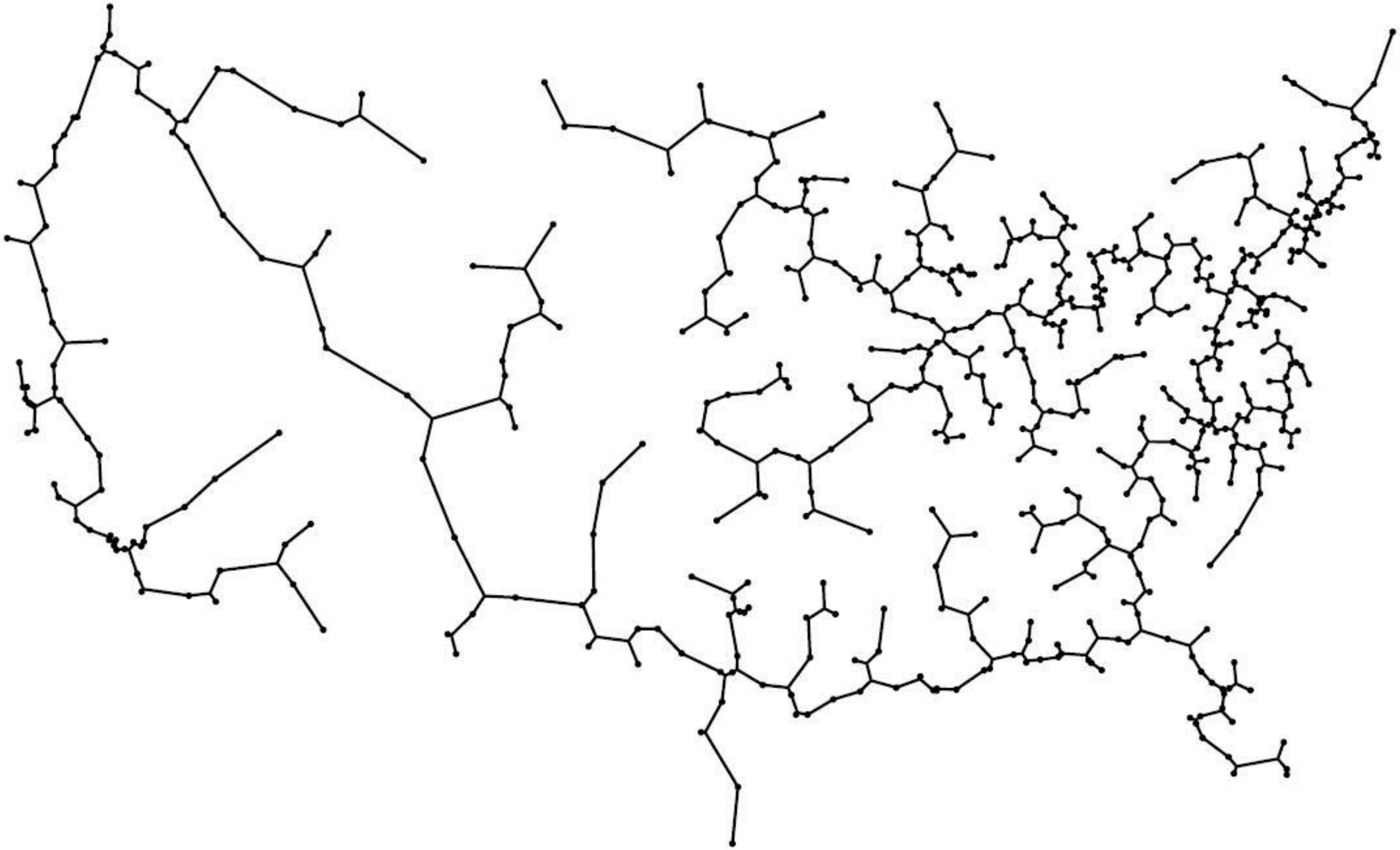
M.R.Garey, **R.L.Graham**, and **D.S.Johnson** proved that the Steiner problem (in the plane) is NP-hard, i.e., most likely there does not exist a polynomial algorithm for solving this problem.

P. Winter, **M.Zachariasen** “Large Euclidean Steiner minimal tree in an hour”, 1996. They created software Geosteiner96. The last version is GeoSteiner 3.1 (it runs under UNIX).

The first versions of the software spent 8 minutes to construct a shortest tree on 100 random points.

D.M.Warme, **P.Winter**, **M.Zachariasen** “Exact algorithms for plane Steiner tree problems: a computational study”, 1998. They made an essential progress:

They stated that their software can construct a shortest tree on 2000 points for reasonable time.



532 cities in the United States

On the cite http://www.diku.dk/hjemmesider/ansatte/martinz/geosteiner/ one can read the following:

Would you like to see a *large* Steiner tree? Here is the optimal solution for the 10000 point Euclidean instance in the OR-Library!

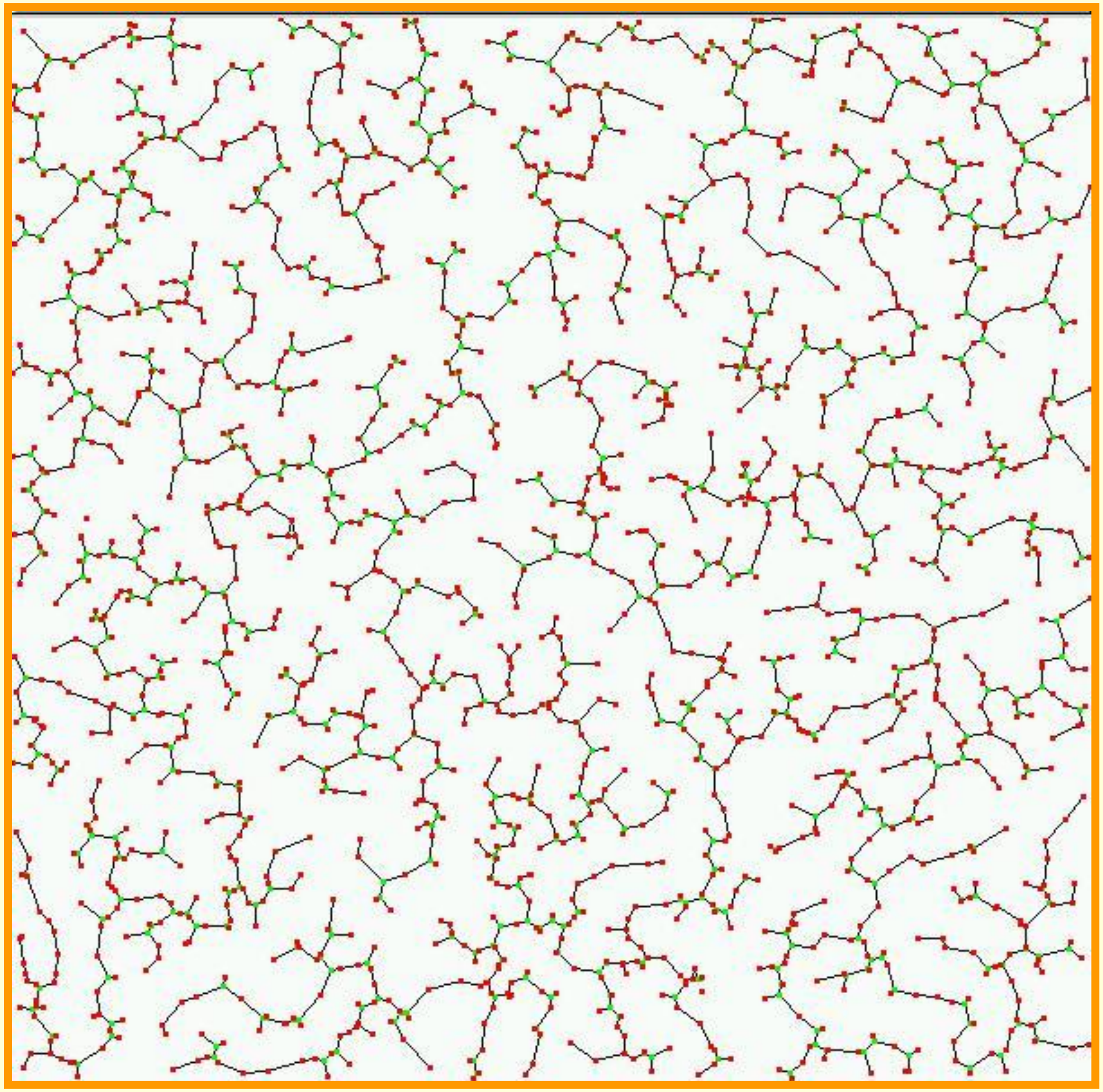
Unfortunately, the reference is broken.

There are some heuristics. For example,

http://cse.taylor.edu/~bbell/

Steiner problem solution (senior project)

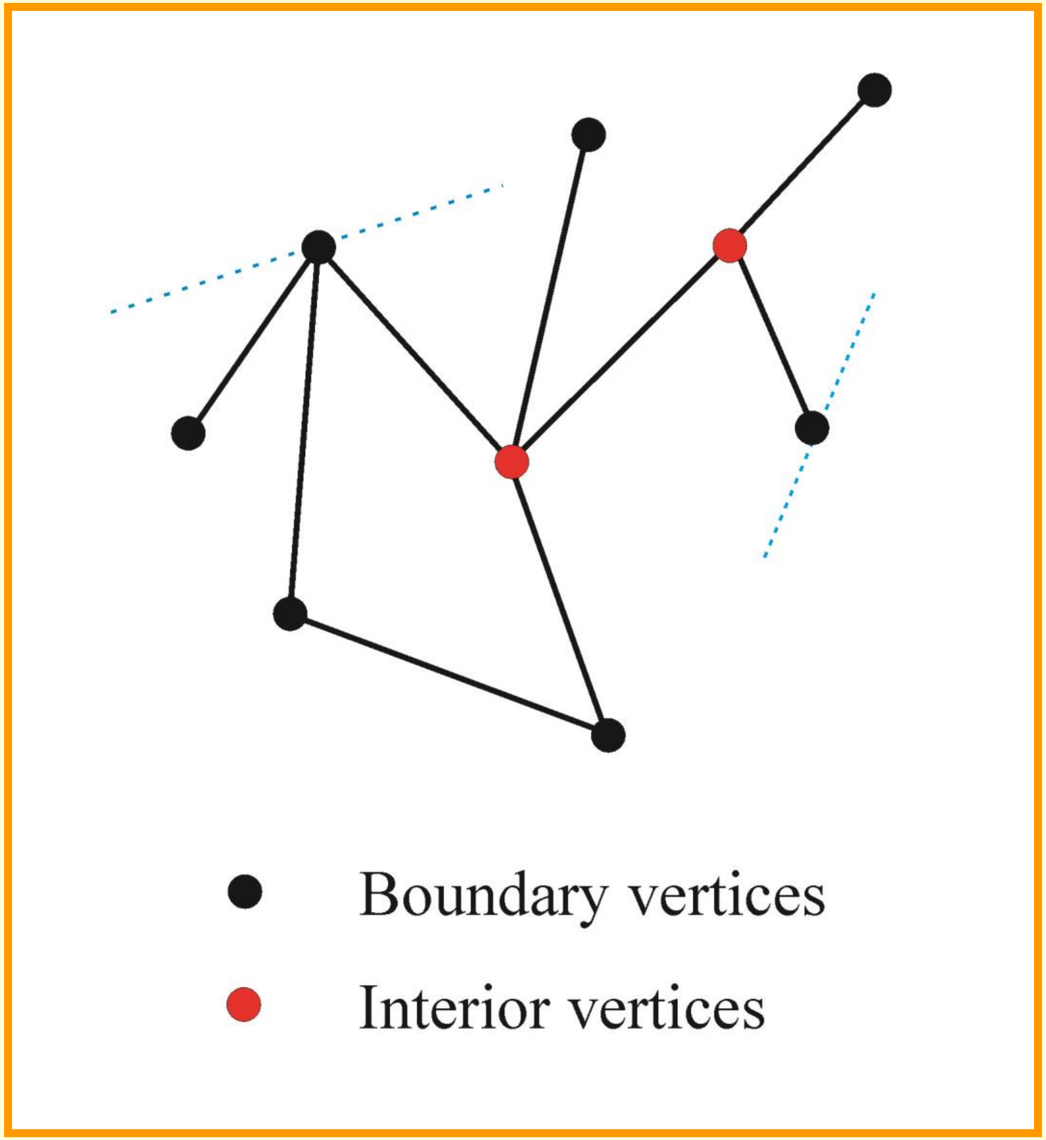
represents an approximate solution of Steiner problem (1999 year, 1600 points).



Methods of Investigation.

(1) Introducing adequate characteristics of the objects in consideration and revealing their relations.

Geometry and topology of plane linear trees.



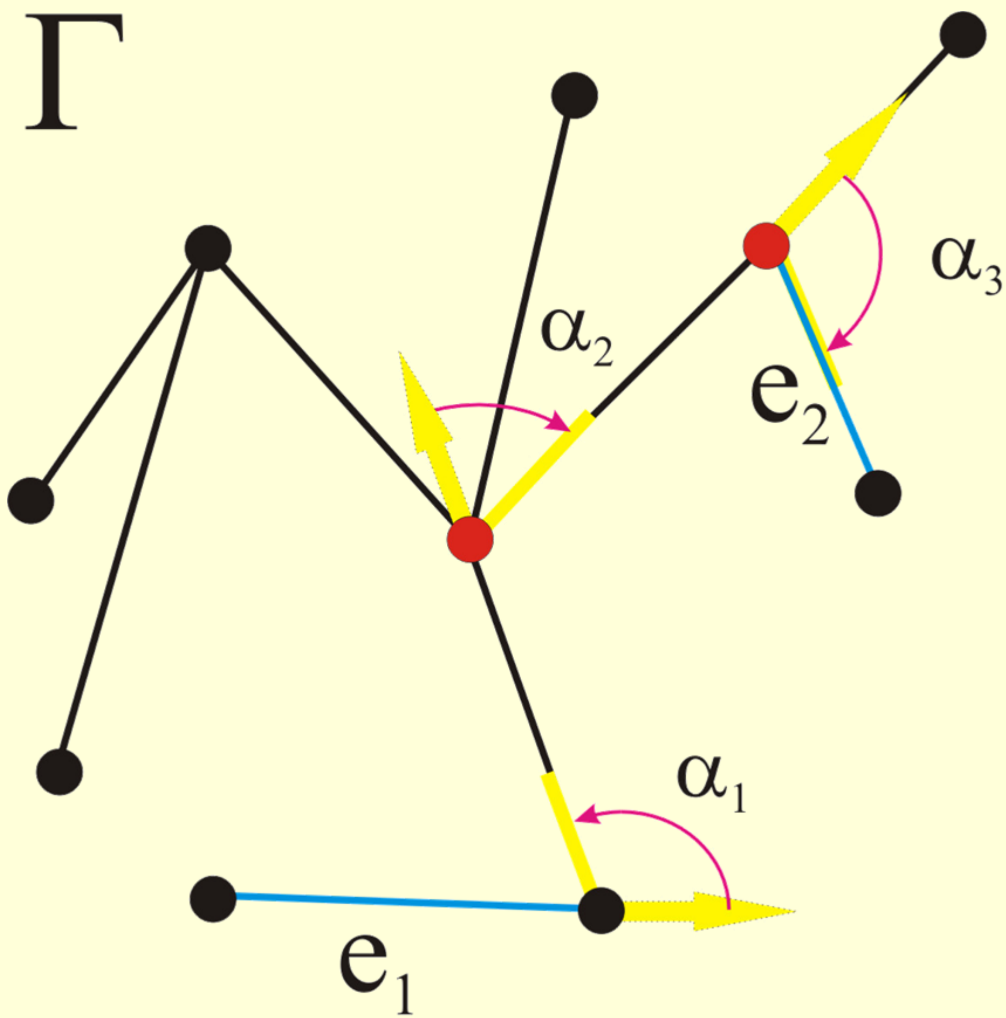
Problem. Find relations between the structure of a linear graph and the geometry of its boundary.

Are there some restrictions on the possible structures of plane graphs setting by geometrical properties of their boundary sets?

In what terms one can describe the structure of linear graph and the geometry of boundary set to reveal a good relation?

Characteristic of graph structure

The twisting number of plane linear trees



$\alpha_1 > 0$

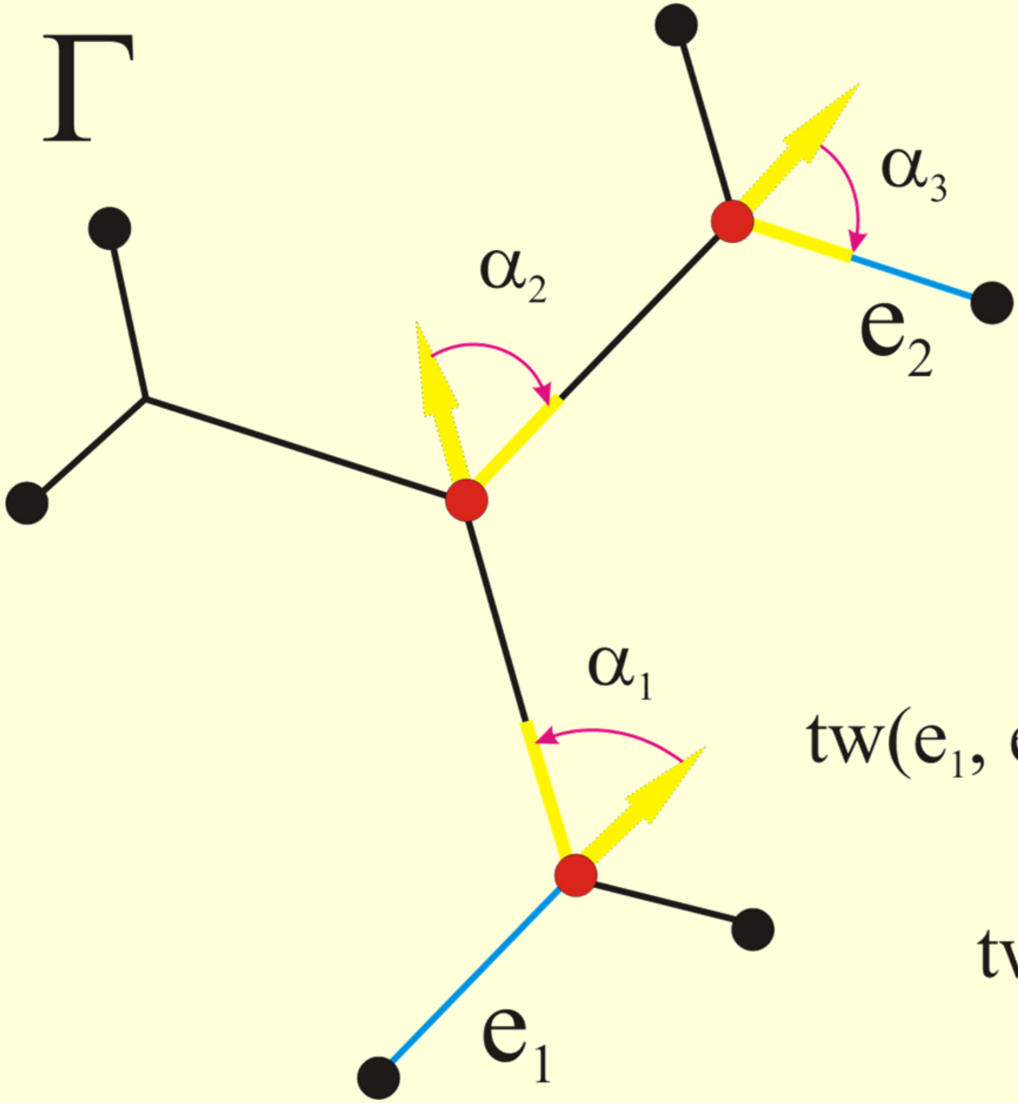
$\alpha_2 < 0$

$\alpha_3 < 0$

$$tw(e_1, e_2) = \frac{3}{\pi} \sum \alpha_i$$

$$tw(\Gamma) = \max \ tw(e_i, e_j)$$

The twisting number of local minimal binary trees



$$\alpha_1 = \frac{\pi}{3} * 1$$

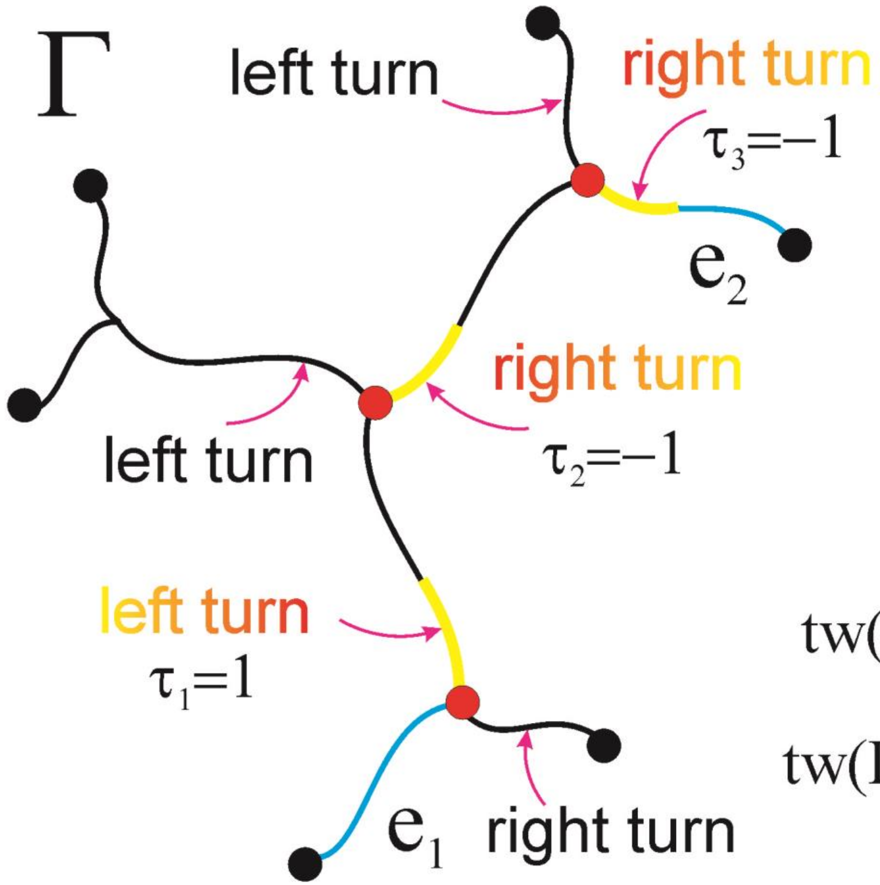
$$\alpha_2 = \frac{\pi}{3} * (-1)$$

$$\alpha_3 = \frac{\pi}{3} * (-1)$$

$$tw(e_1, e_2) = \frac{3}{\pi} \sum \alpha_3 = 1 + (-1) + (-1)$$

$$tw(\Gamma) = \max \ tw(e_i, e_j)$$

The twisting number of plane binary trees

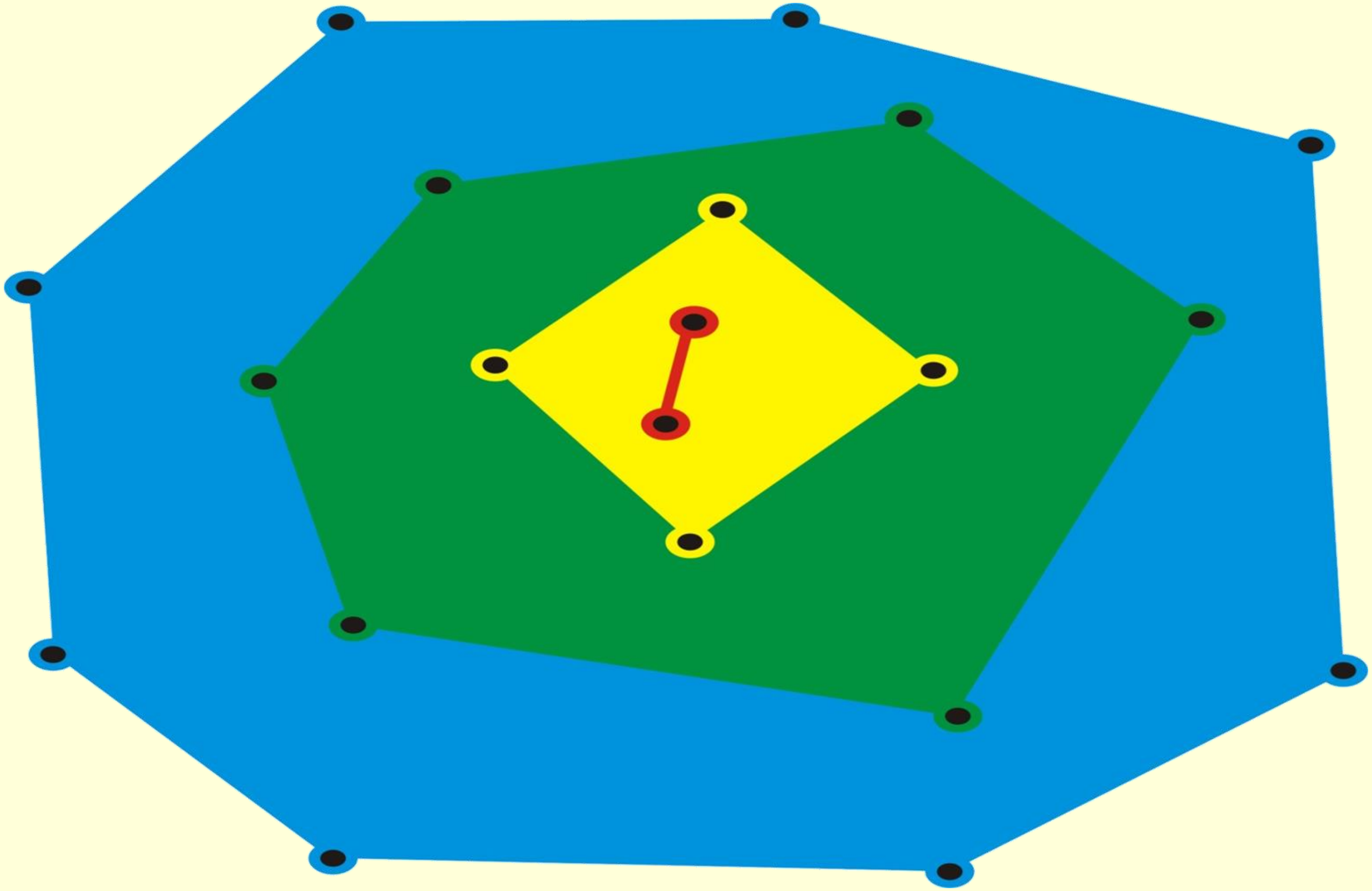


$$tw(e_1, e_2) = \tau_1 + \tau_2 + \tau_3$$

$$tw(\Gamma) = \max\ tw(e_i, e_j)$$

Characteristic of boundary set geometry

Convexity levels



This set has four convexity levels

Number of convexity levels and twisting number

Theorem (A.Ivanov, A.Tuzhilin). Let Γ be a linear plane tree and n the number of convexity levels of its boundary. Then

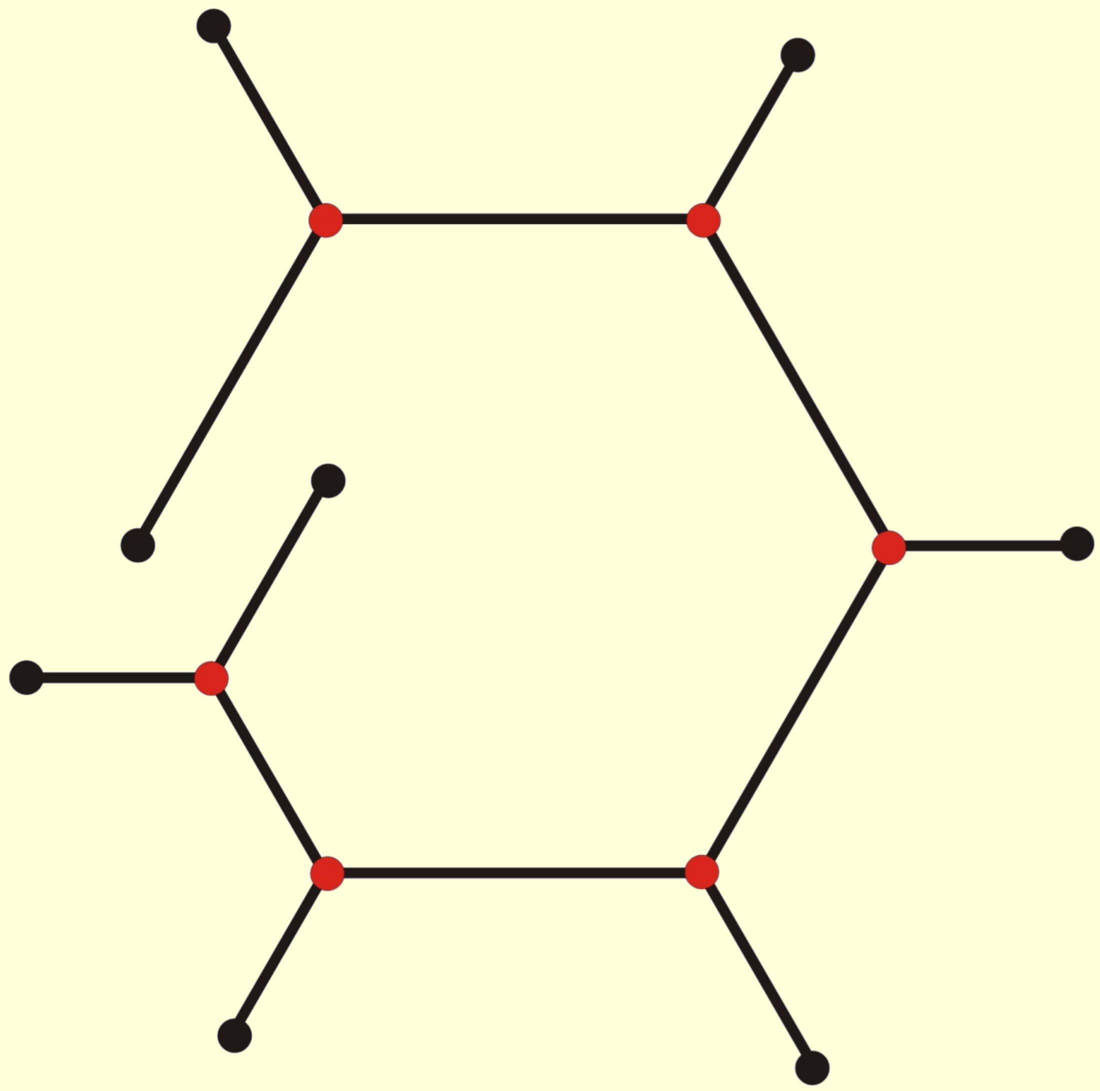
$$tw(\Gamma) \le 12(n-1) + 6.$$

Corollary. Let Γ be a local minimal plane binary tree and n the number of convexity levels of its boundary. Then

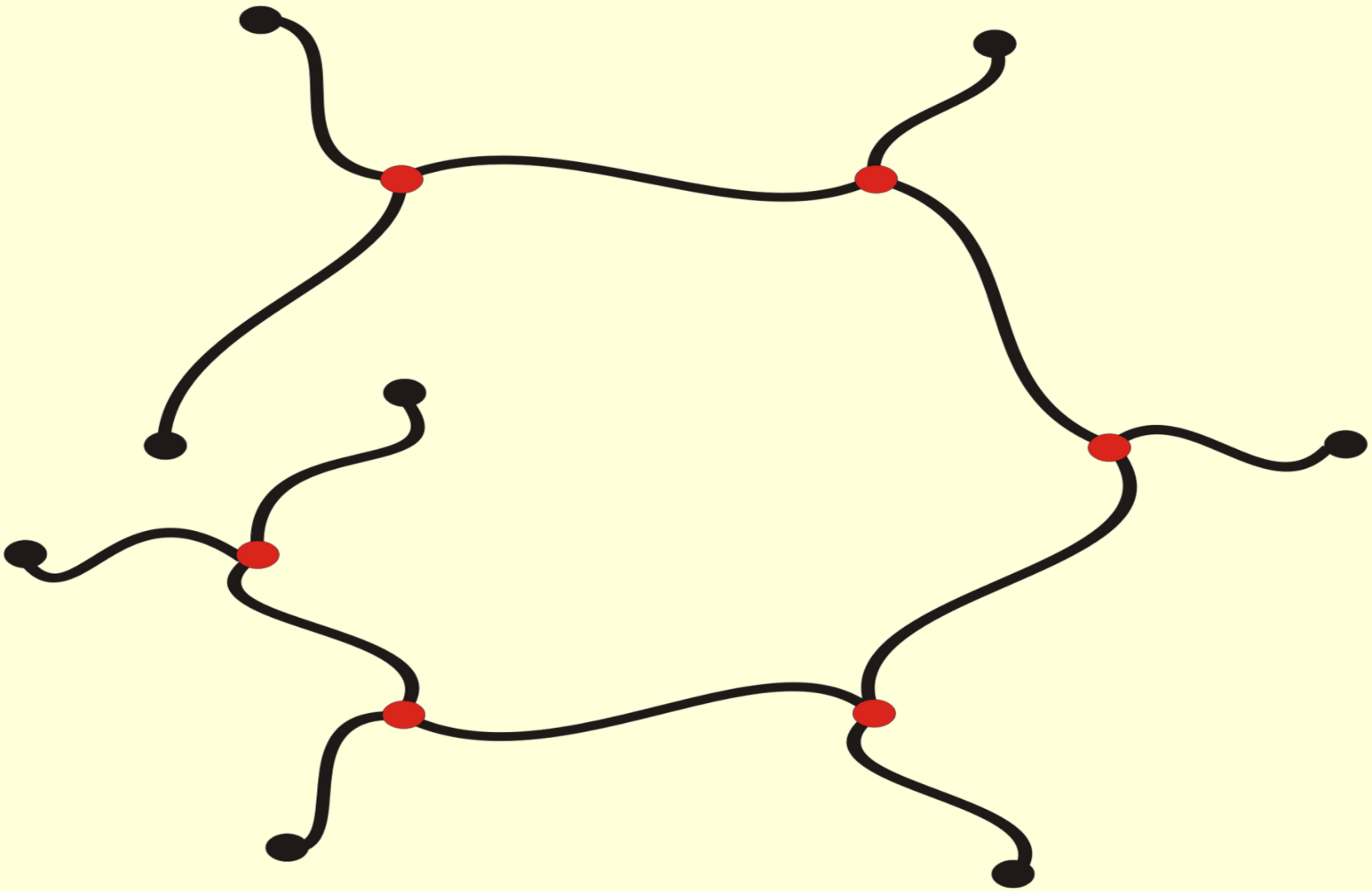
$$tw(\Gamma) \le 12(n-1) + 5.$$

The boundary set consisting of just one convexity level we call convex.

It is impossible to deform this local minimal tree by changing its edges lengths to obtain the one with a convex boundary and without self-intersections.



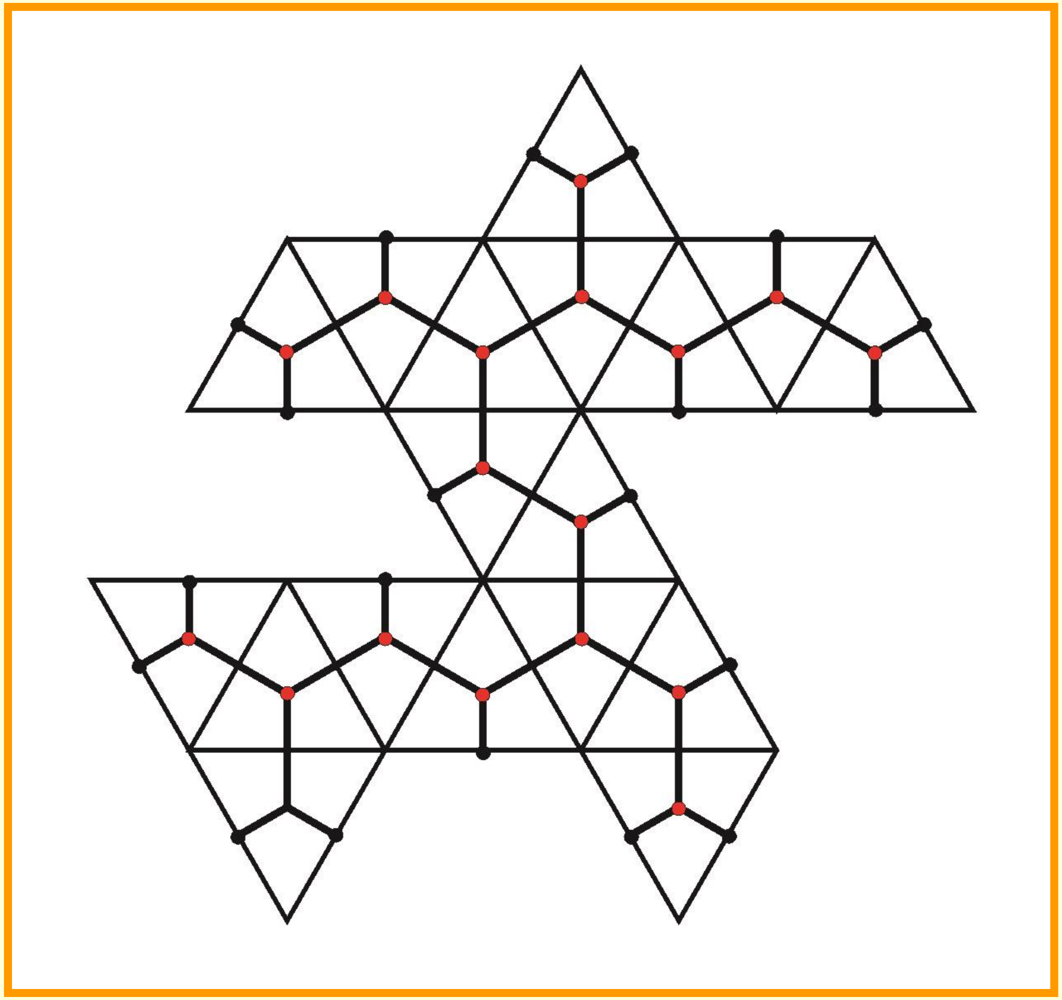
There does not exist a local minimal binary tree with a convex boundary, such that it is planar equivalent to the tree depicted below.



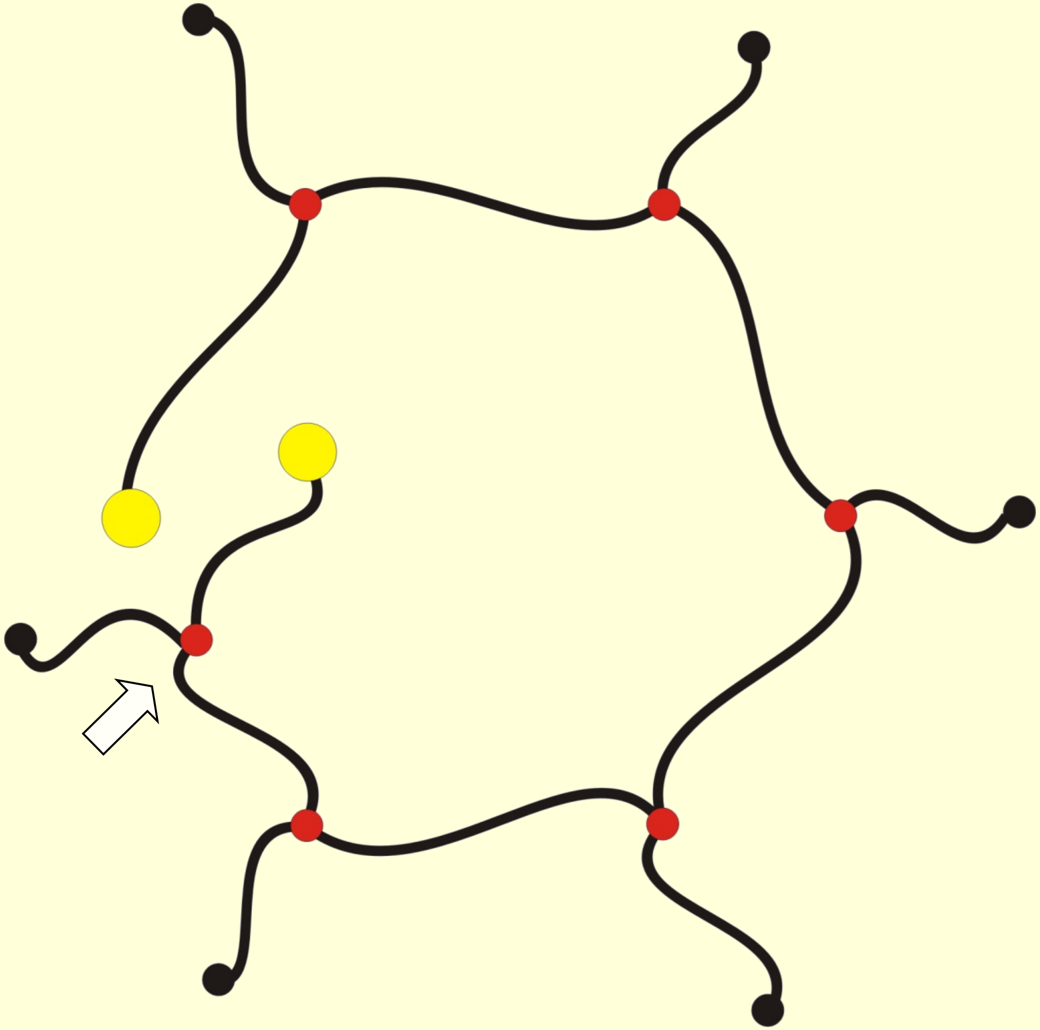
(2) Passing from one mathematical language to another one.

Local minimal trees joining the vertices of convex polygons.

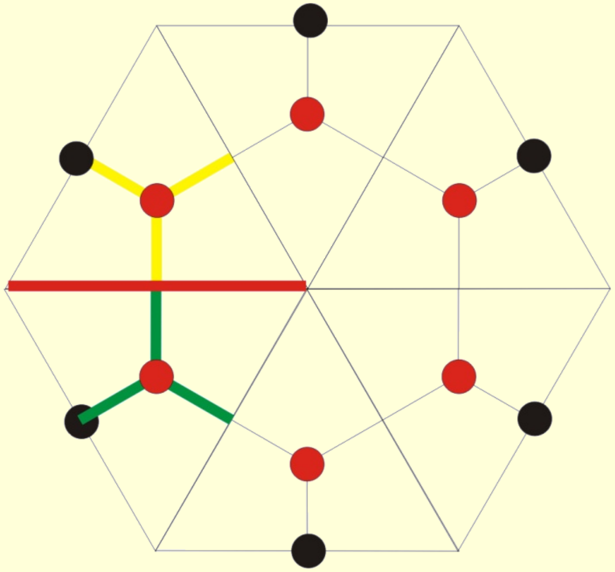
1) Dual language of tilings



Which binary trees are dual graphs of tilings?



This binary tree can not be realized as the dual graph of a tiling.

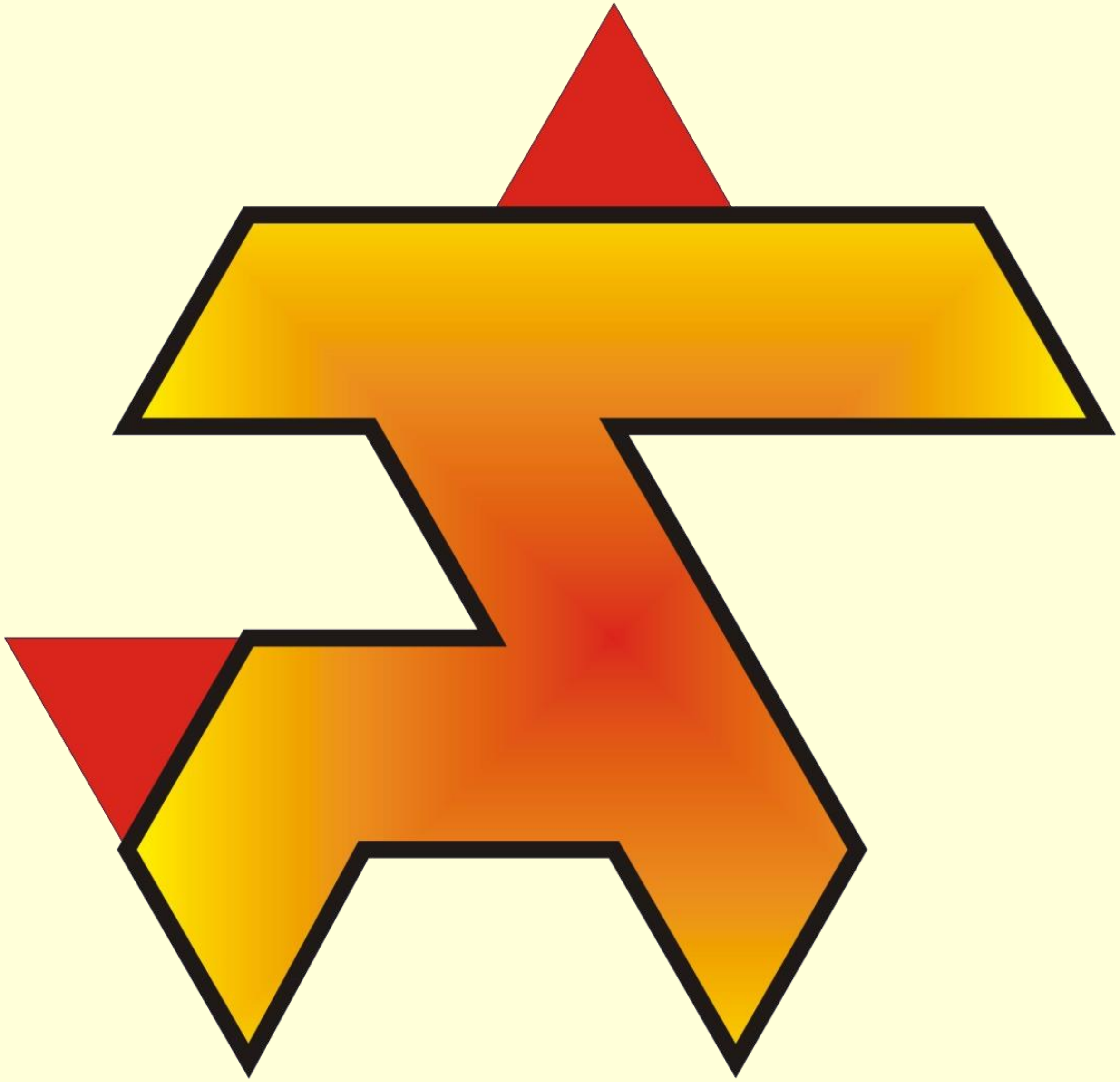


Tiling realization

Theorem (A.Ivanov, A.Tuzhilin). If the twisting number of a plane binary tree does not exceed 5, then it can always be realized as the dual graph of a tiling.

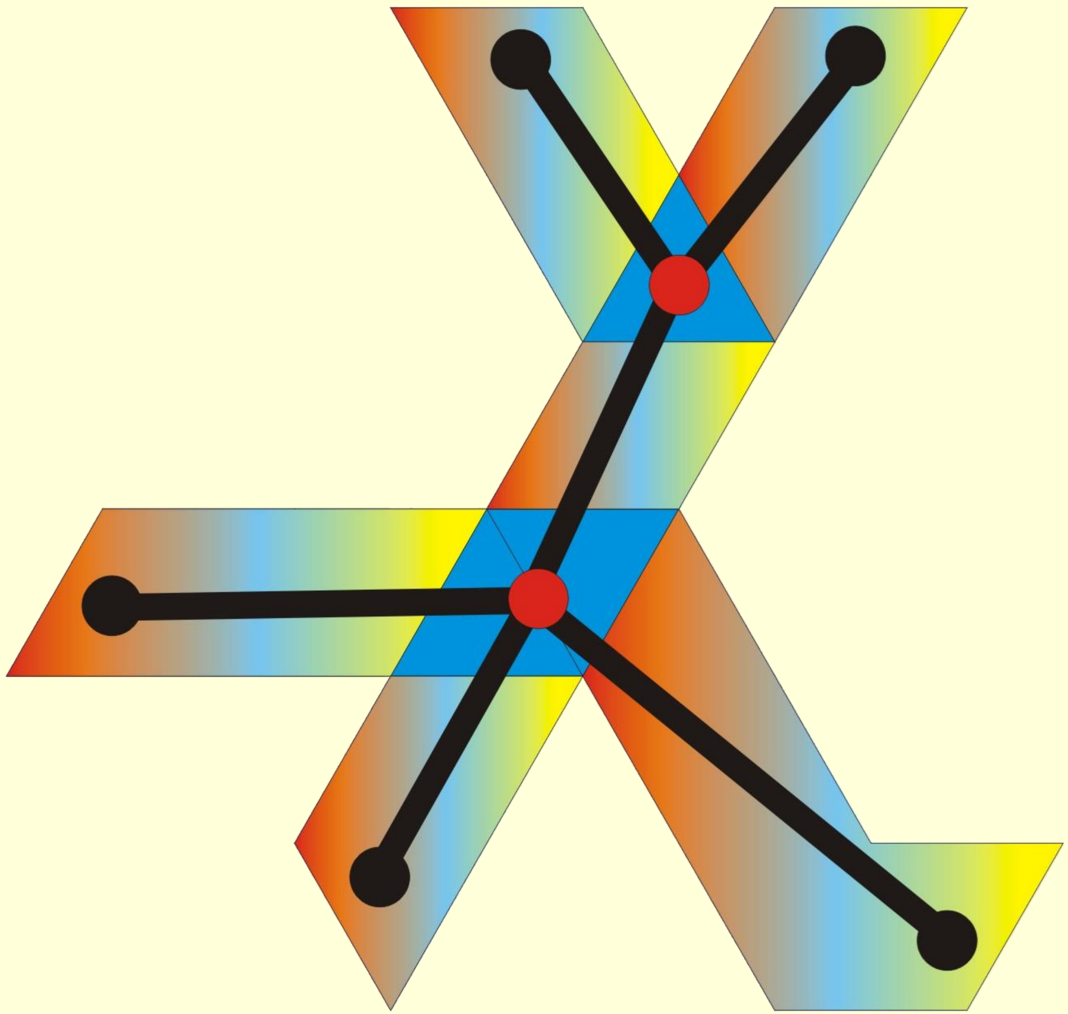
Skeleton is a tiling without growths

(so, we constructed a decomposition into skeleton and growths)



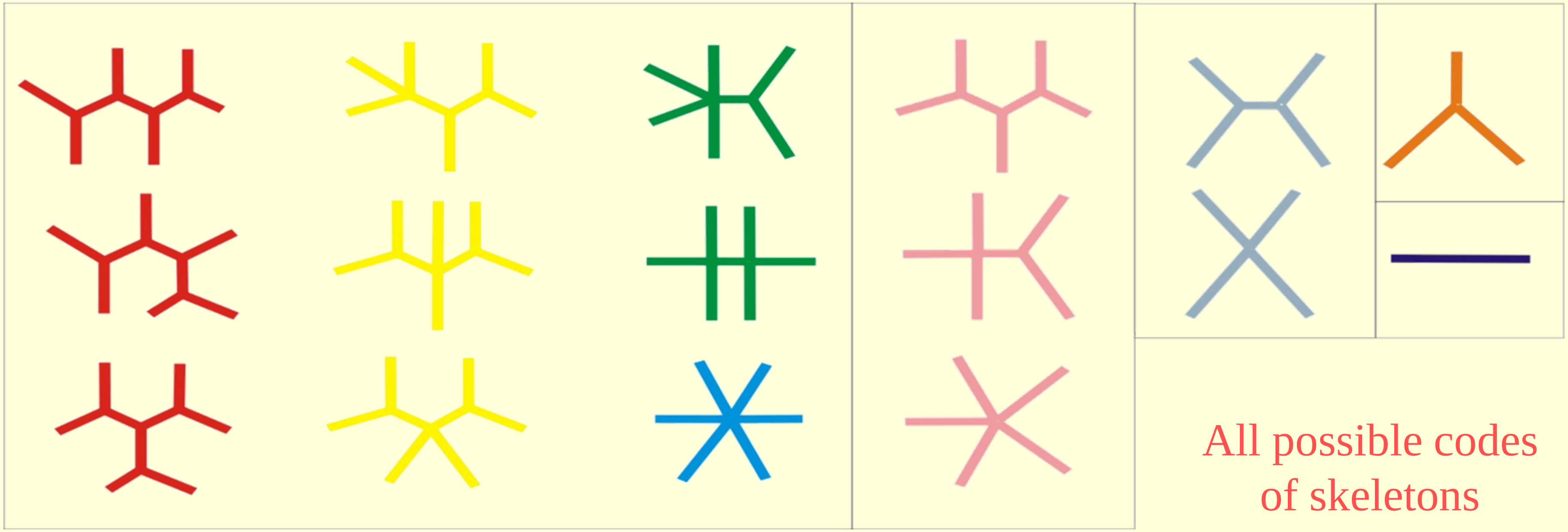
Classification of skeletons

Code of a skeleton



Topological classification of skeletons: codes

Theorem (A.Ivanov, A.Tuzhilin). Consider all skeletons whose dual graphs twisting numbers are at most 5 and for each of these skeletons construct its code. Then, up to planar equivalence, we obtain all plane graphs with at most 6 vertices of degree 1 and without vertices of degree 2. In particular, every such skeleton contains at most 4 branching points and at most 9 linear parts.



All possible codes of skeletons

Criterion of convex minimal realization

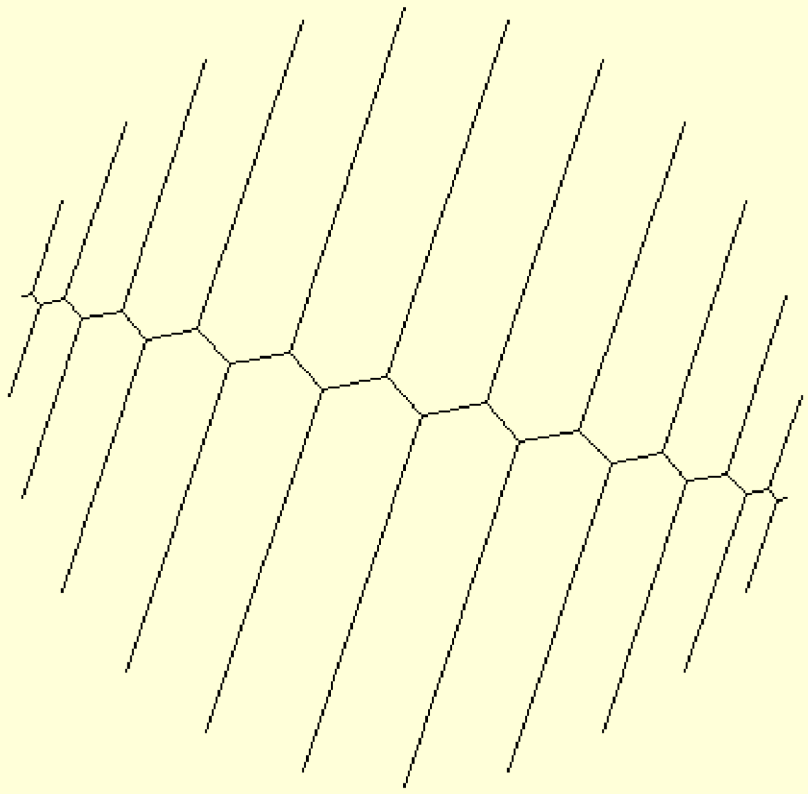
Theorem (A.Ivanov, A.Tuzhilin). If the twisting number of a plane binary tree G does not exceed 5, then there exists a local minimal binary tree planar equivalent to G whose boundary is the set of vertices of a convex polygon.

Corollary. A plane binary tree is planar equivalent to a local minimal tree with a boundary consisting of vertices of a convex polygon if and only if the twisting number of the tree is at most 5.

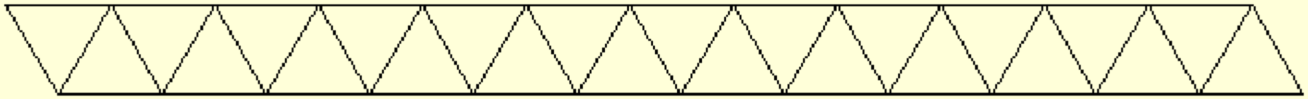
Remark. A.Ivanov and **A.Tuzhilin** have obtained a complete description of all tilings whose twisting numbers are at most 5 (not only the topology of their skeletons, but the geometry of the skeletons and possible growths attachment). This gave complete classification of local minimal binary trees with convex boundary.

(3) Classification.

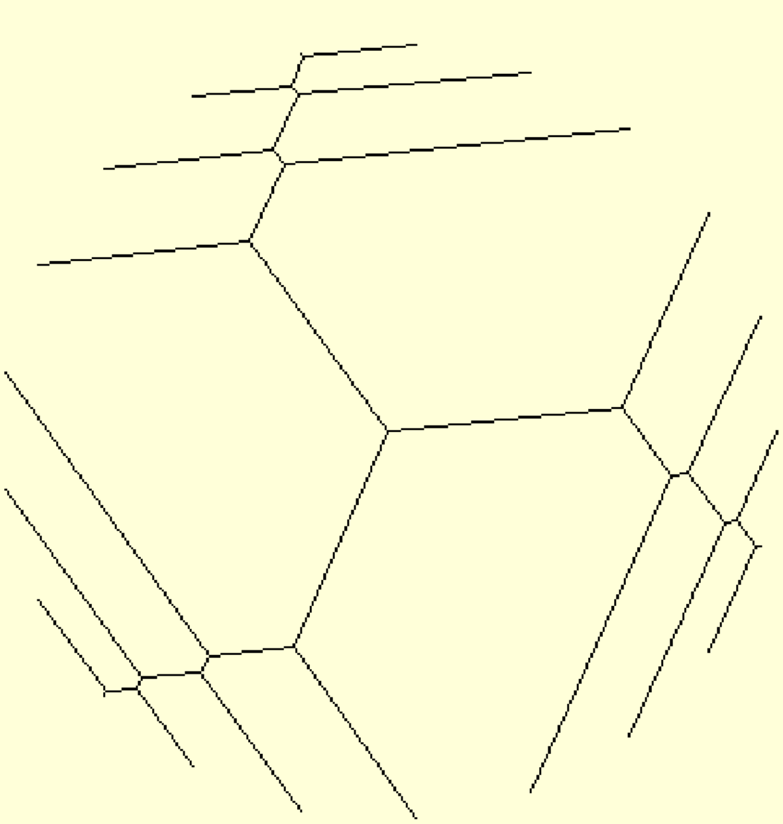
Complete classification of local minimal binary trees of skeleton type joining the vertices of regular n-gons

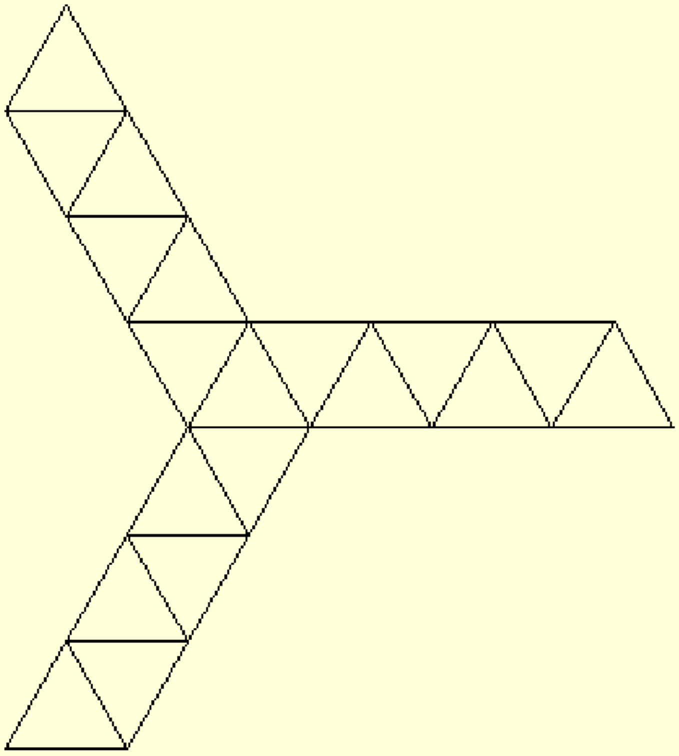


The tree of the snake type exists for any n.

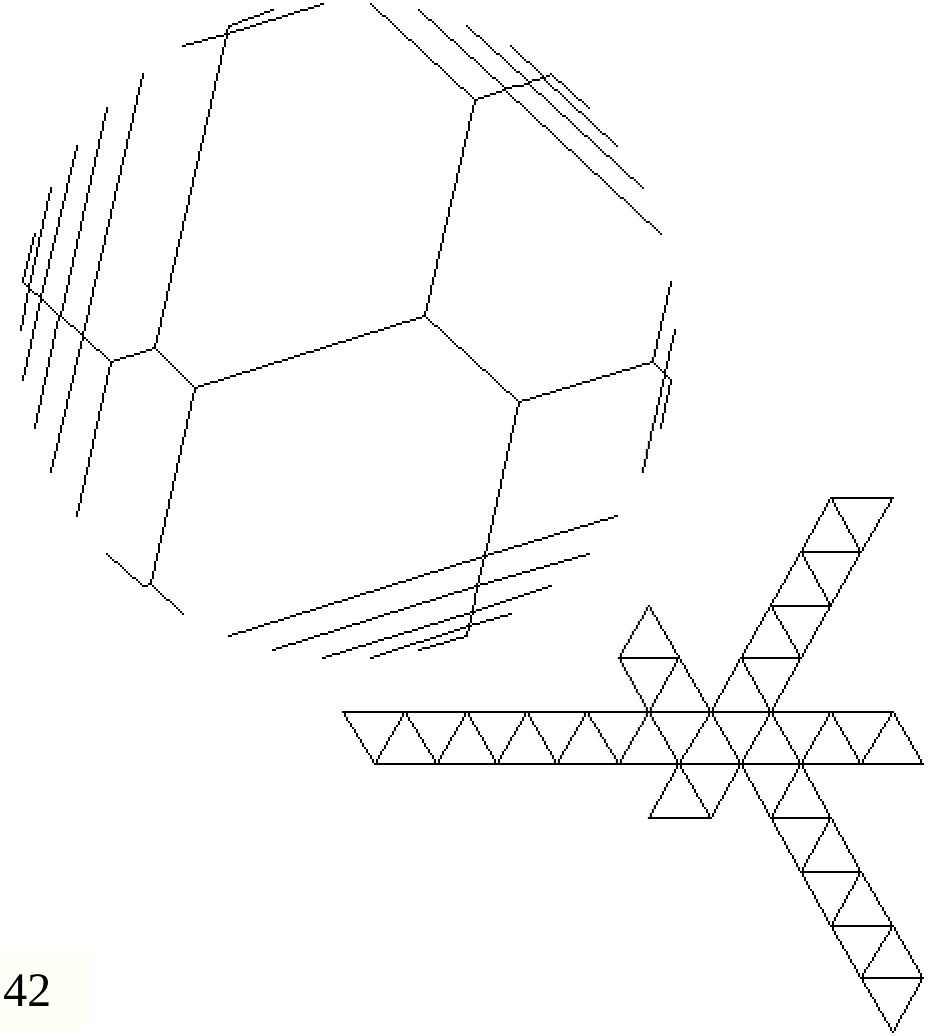


The tree of the type T-joint exists just for n = 6k+3.



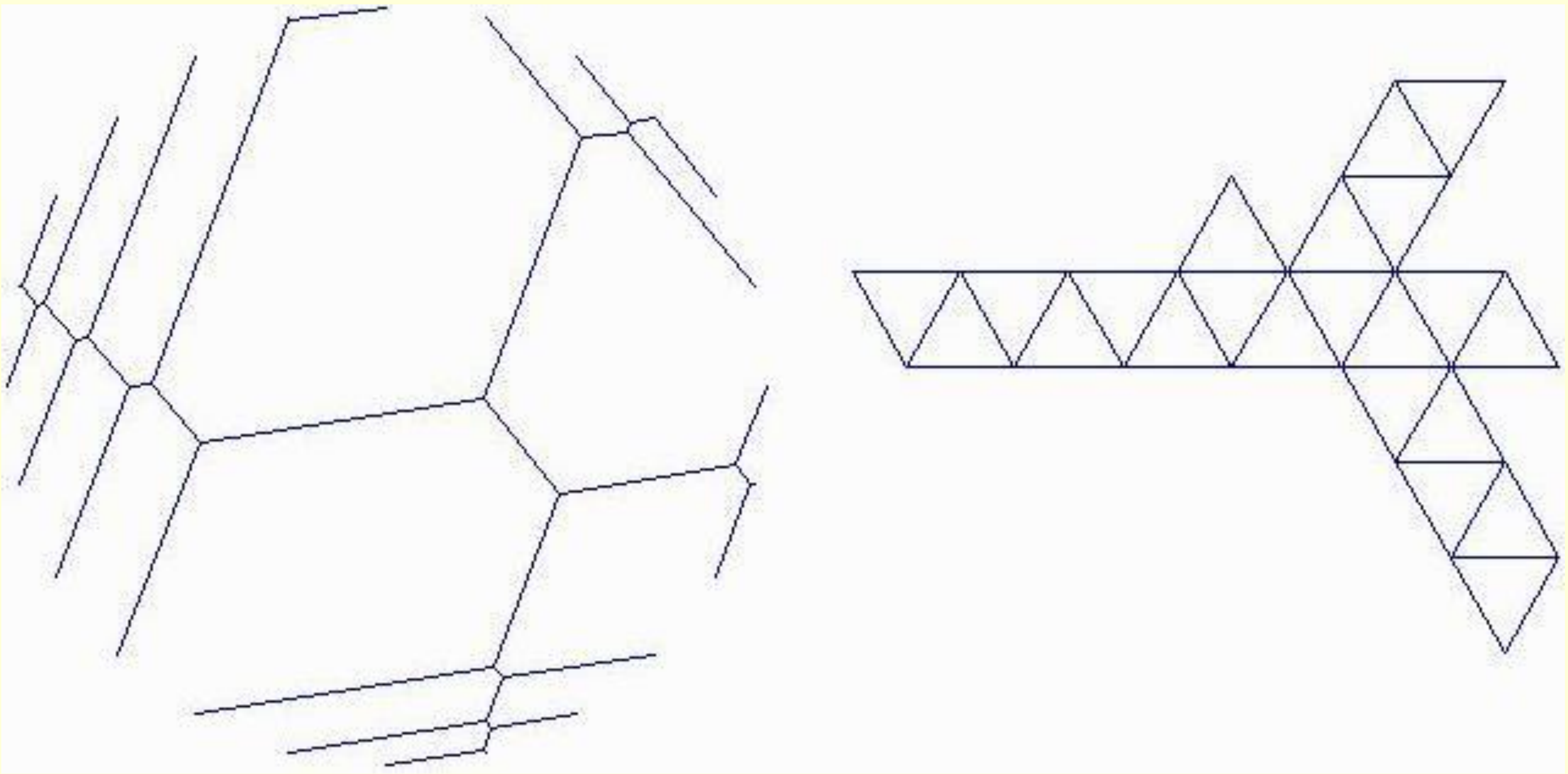


The tree of the type 6-fold exists just for four values of n: 24, 30, 36, 42.



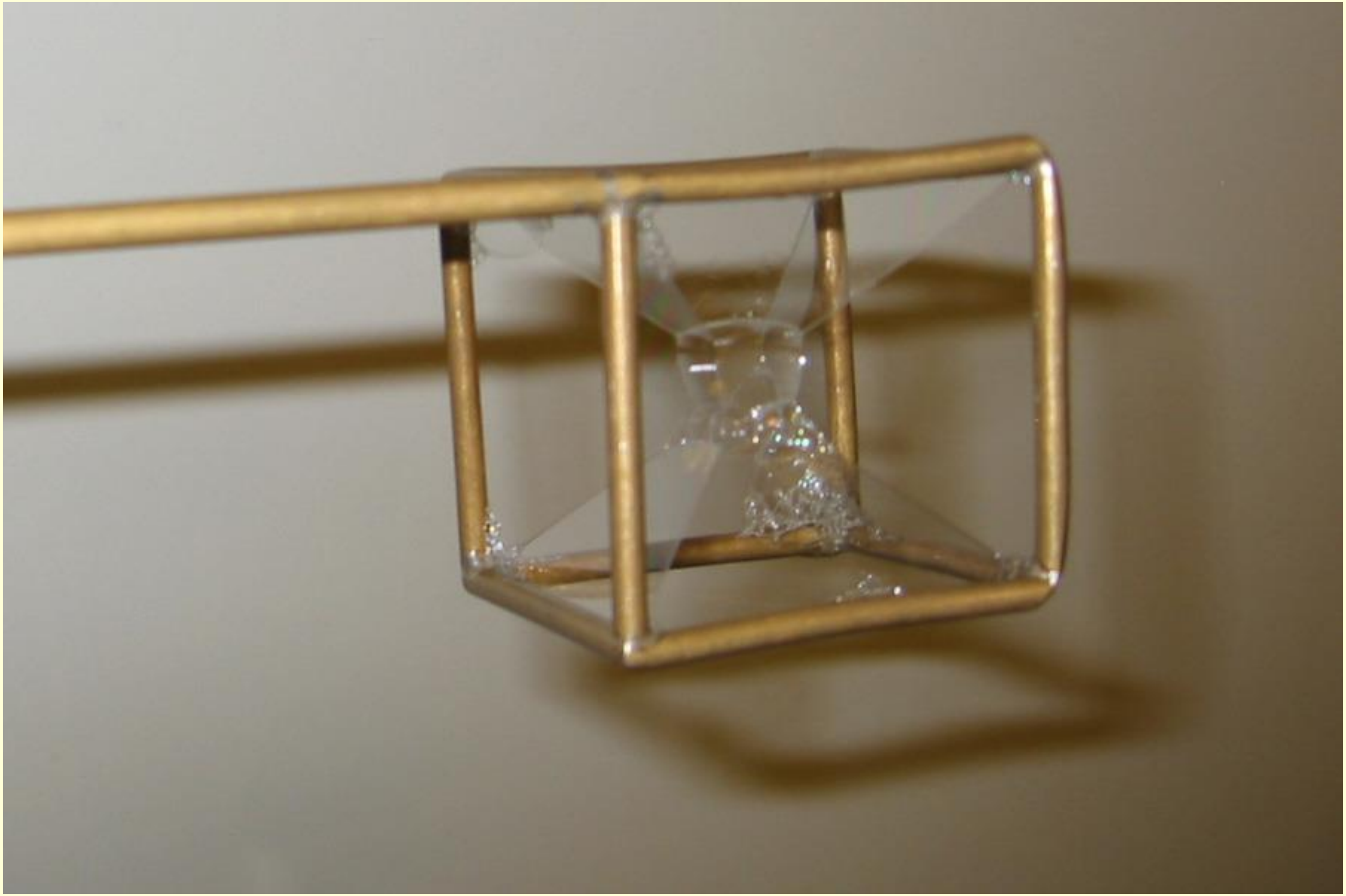
n = 42

The general classification is not completed. A few examples.



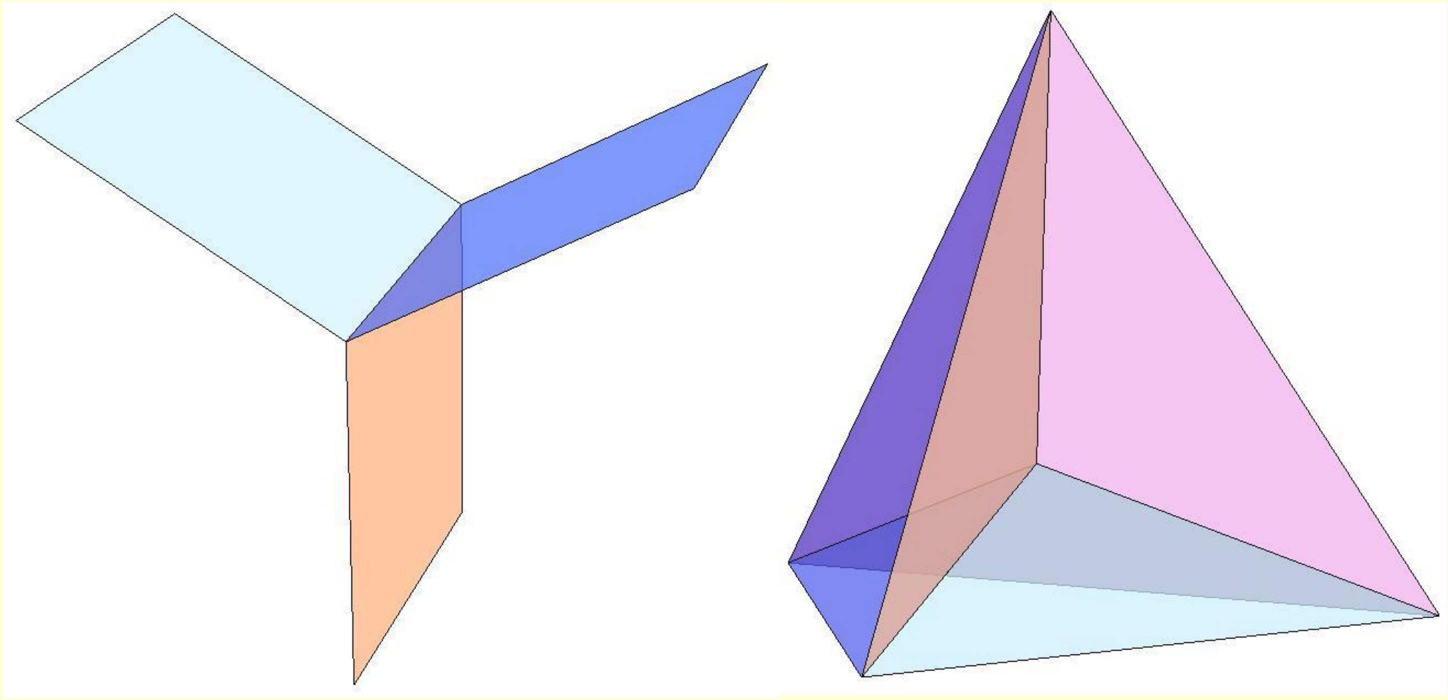
(4) Reduction from a complicated object to a simpler one.

Singularities of stable minimal surfaces (soap films)

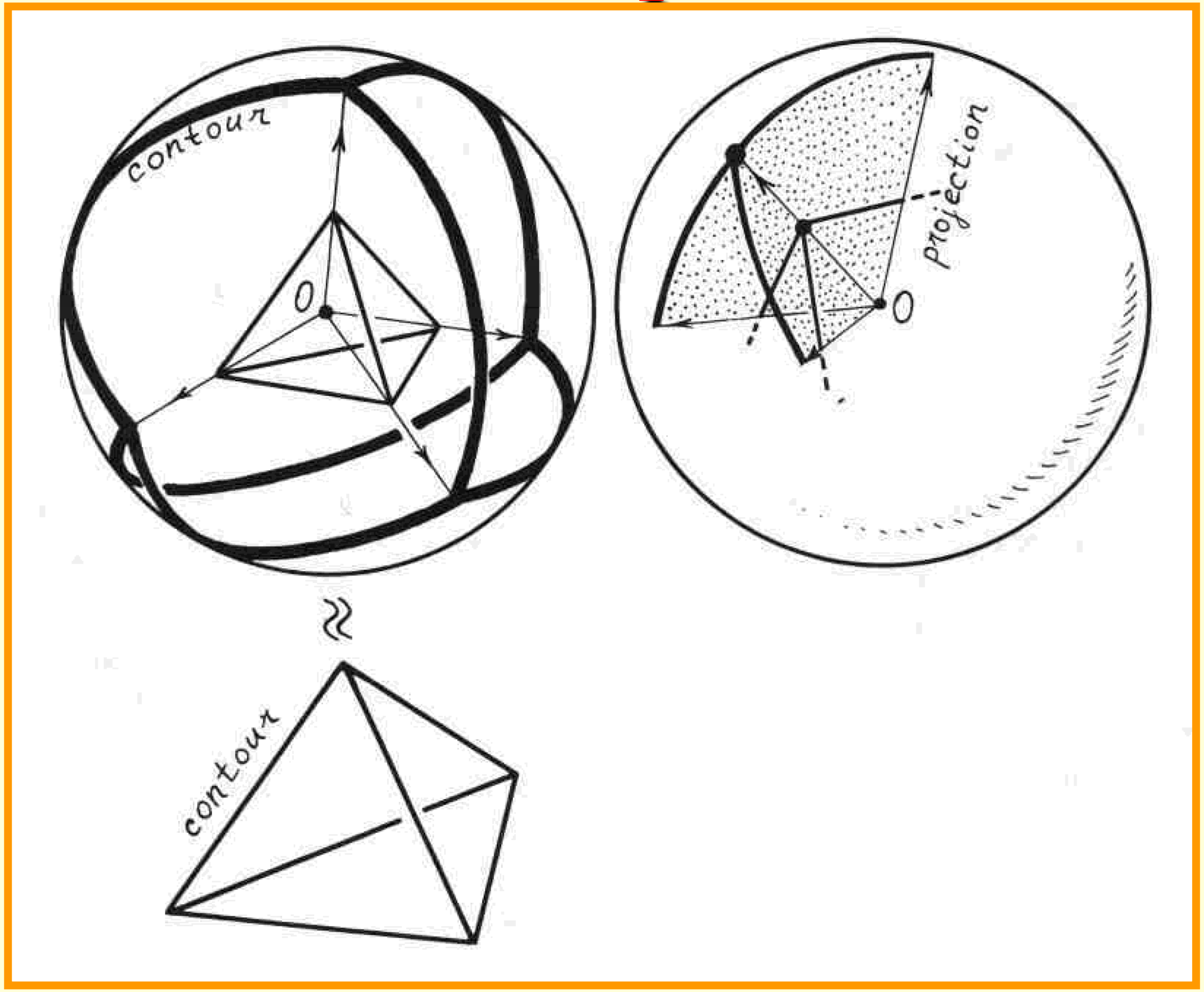


Plateau principles

J.Plateau (1801-1883) formulated four principles, which describe possible singularities on soap films (stable minimal surfaces).

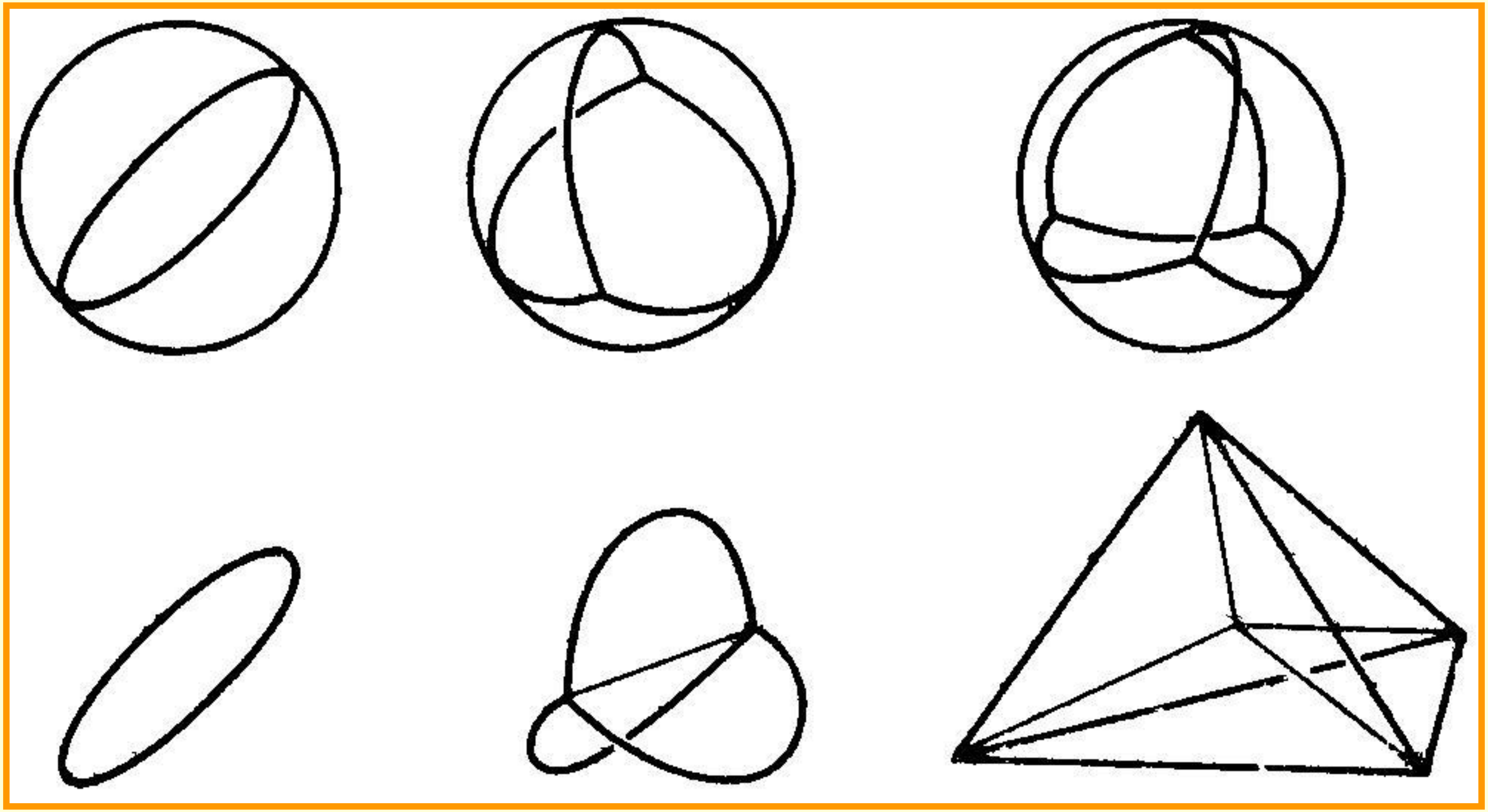


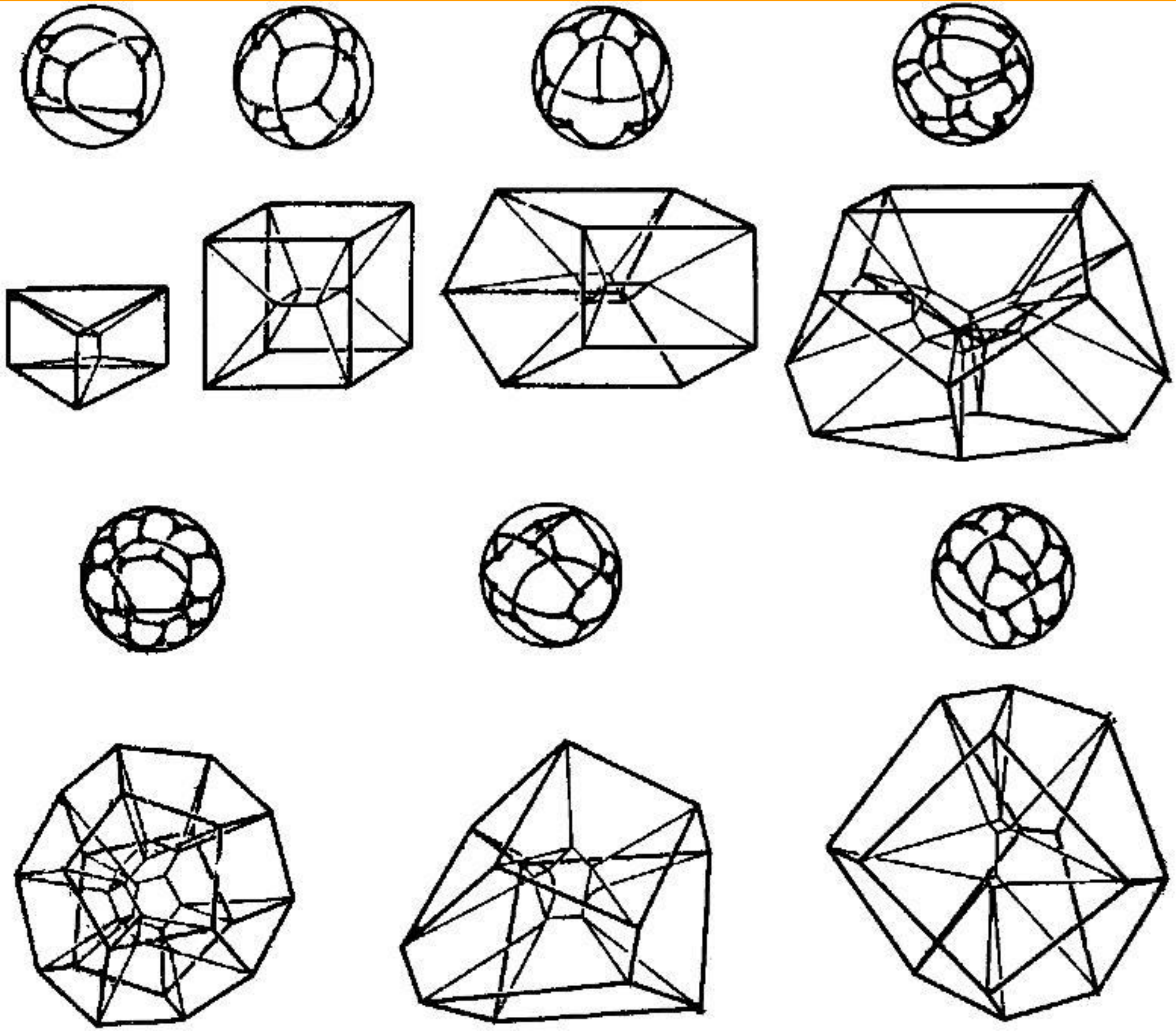
How can one prove that?

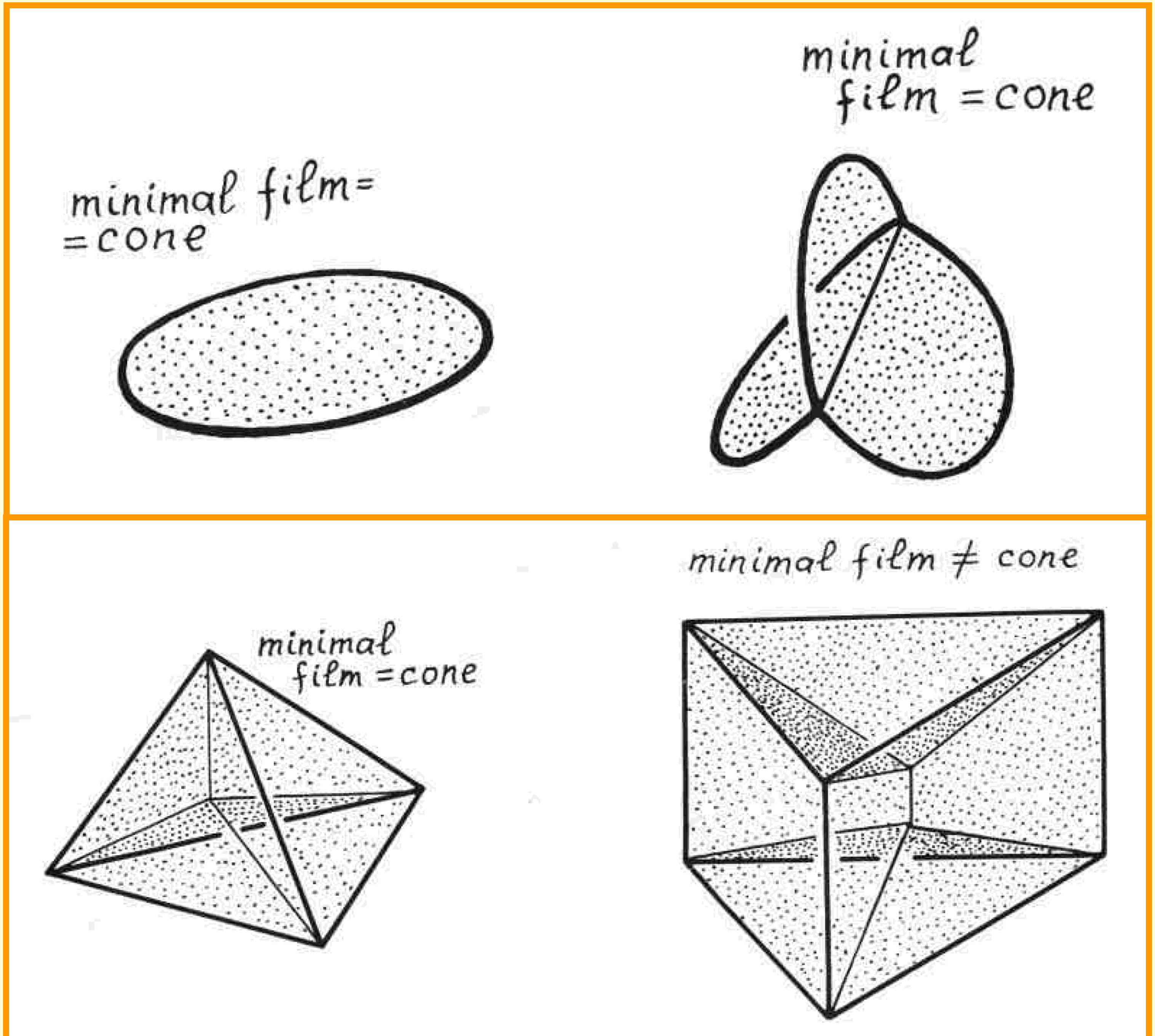


The limiting network minimizes the length locally (each its sufficiently small part is shortest). Such networks are called local minimal.

Ten possible local minimal networks on standard sphere and corresponding soap films (A. Heppes, 1964)

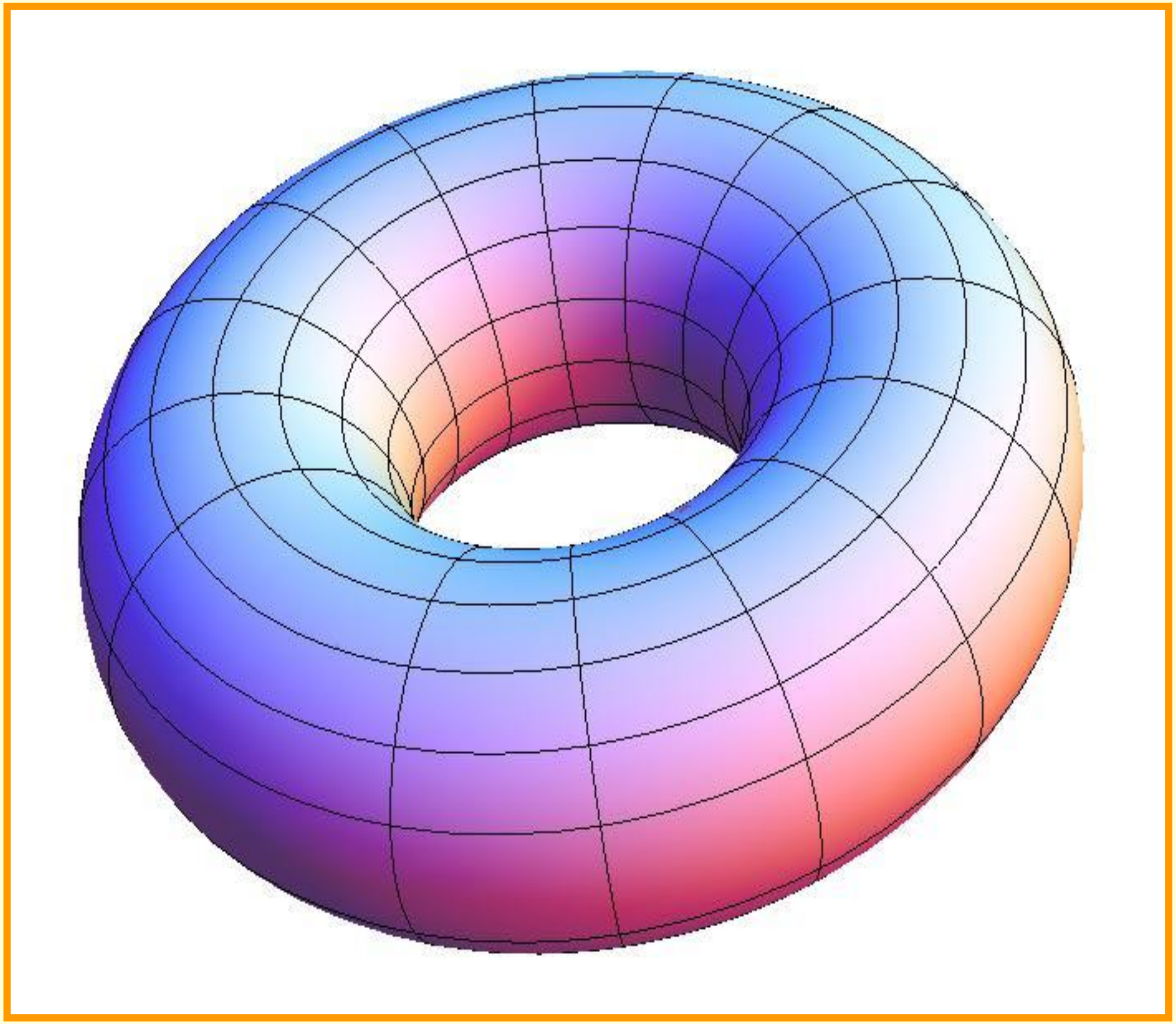




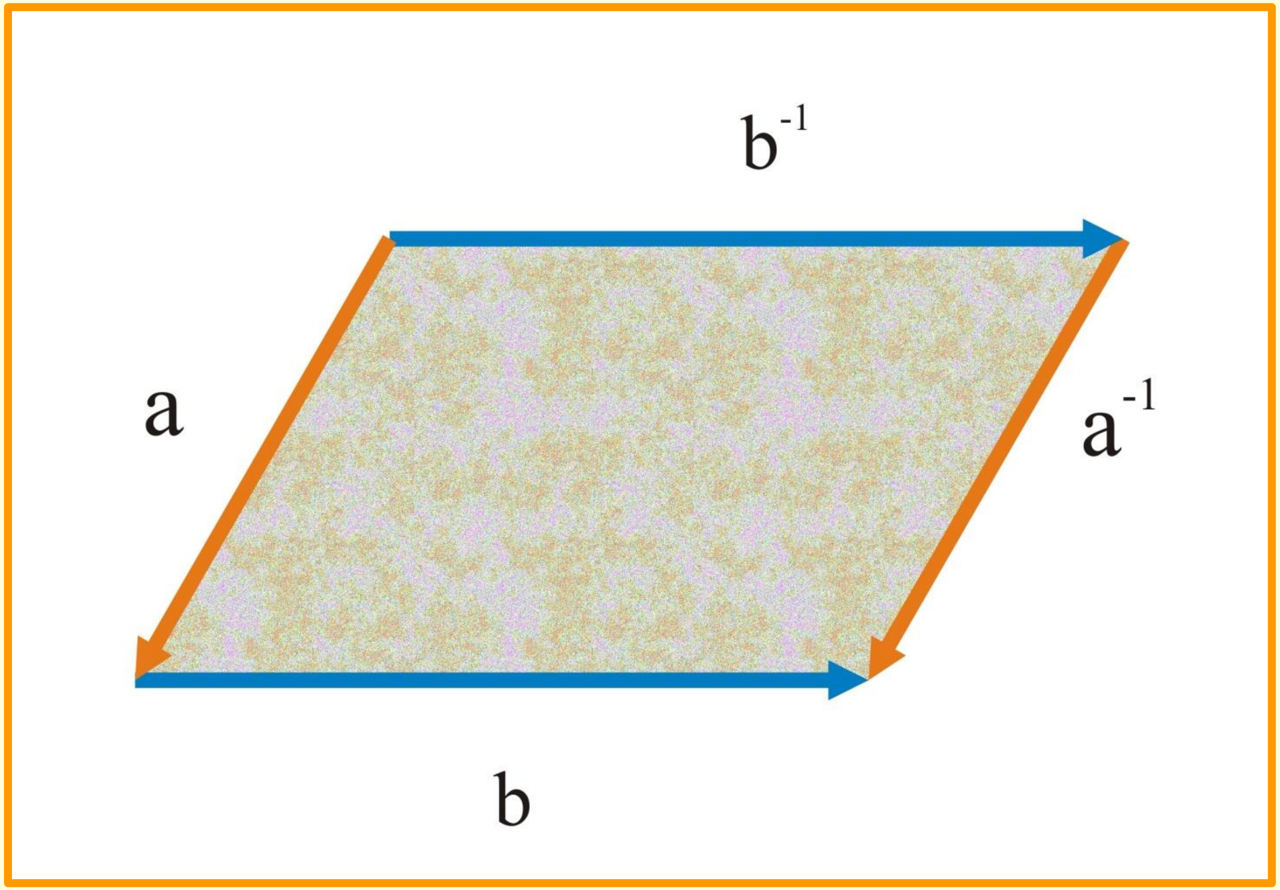

minimal film=
=cone
minimal
film =cone
minimal
film =cone
minimal film ≠ cone

(5) Encoding objects.

Torus T^2

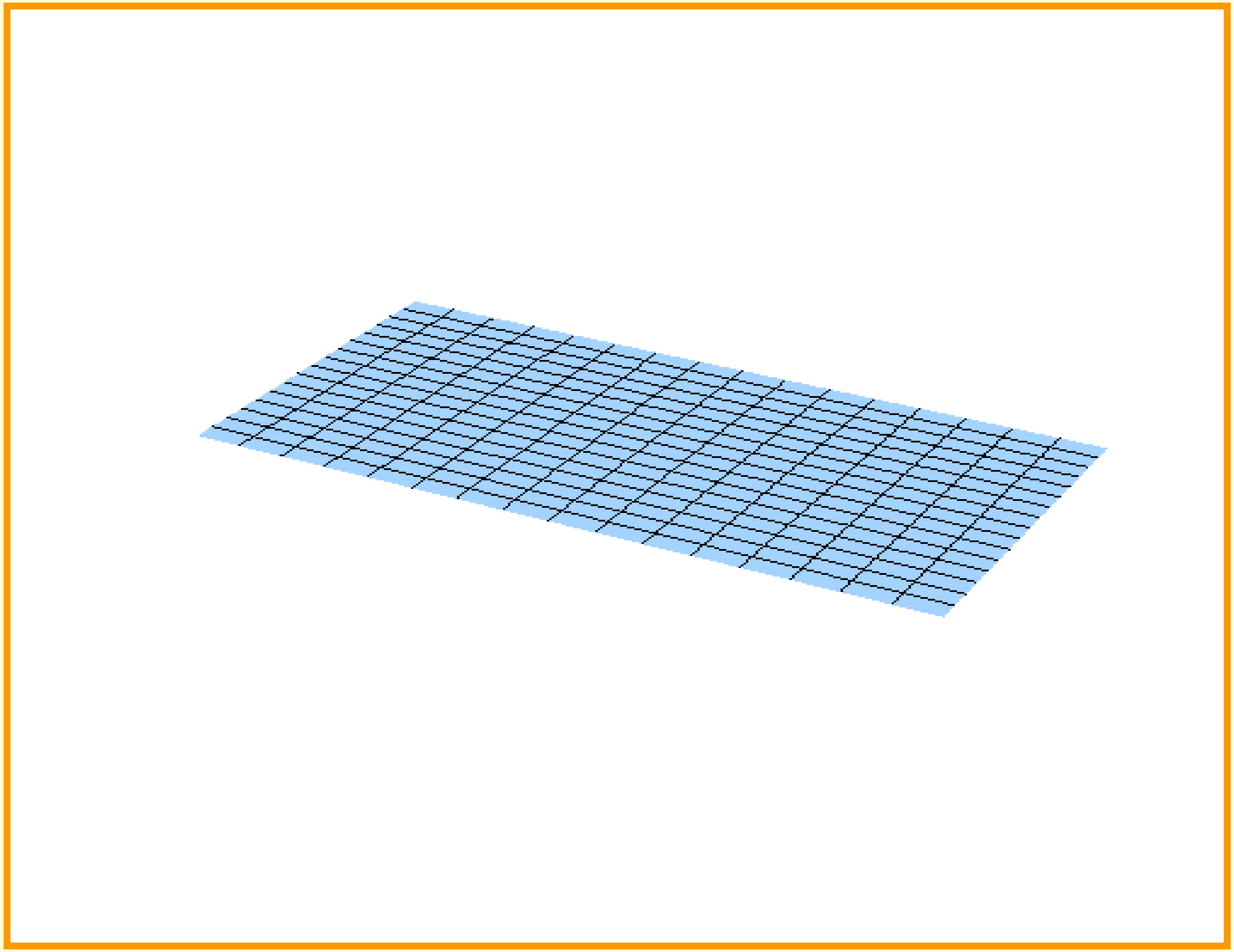


We glue tori T^2 from parallelograms

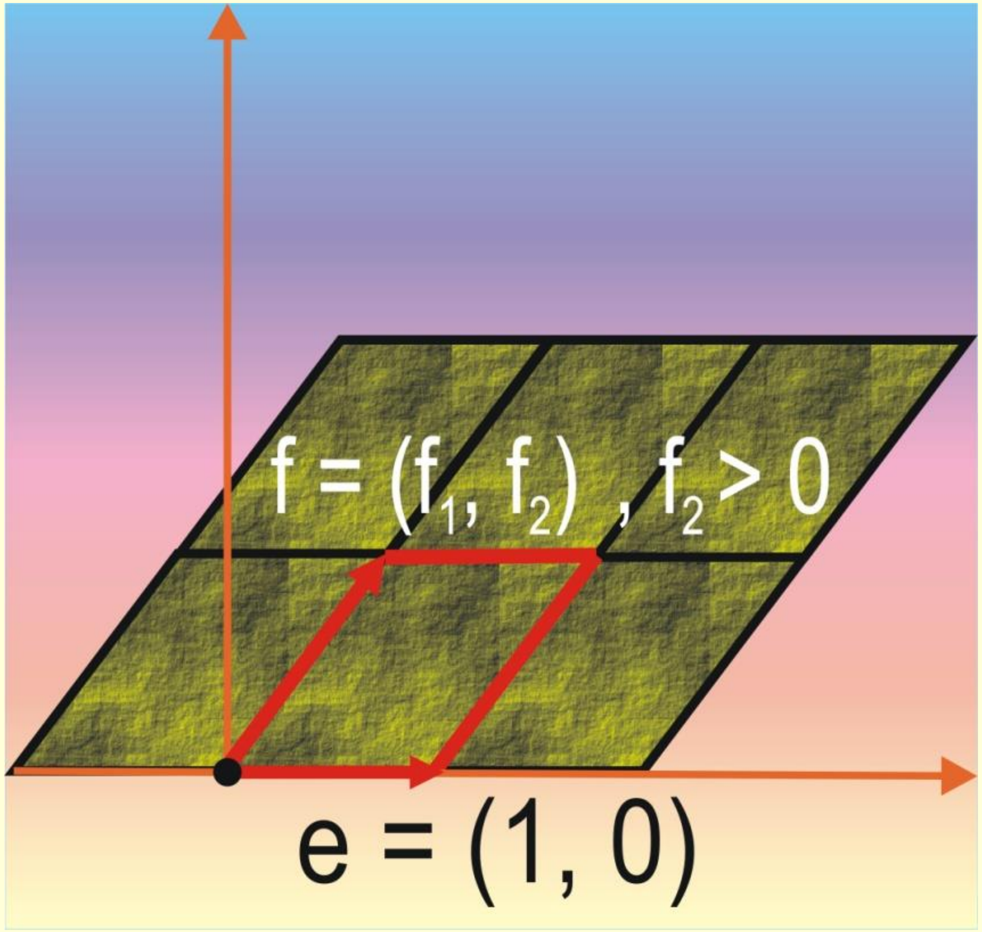


Since parallelogram lies in the plane, we call such tori flat.

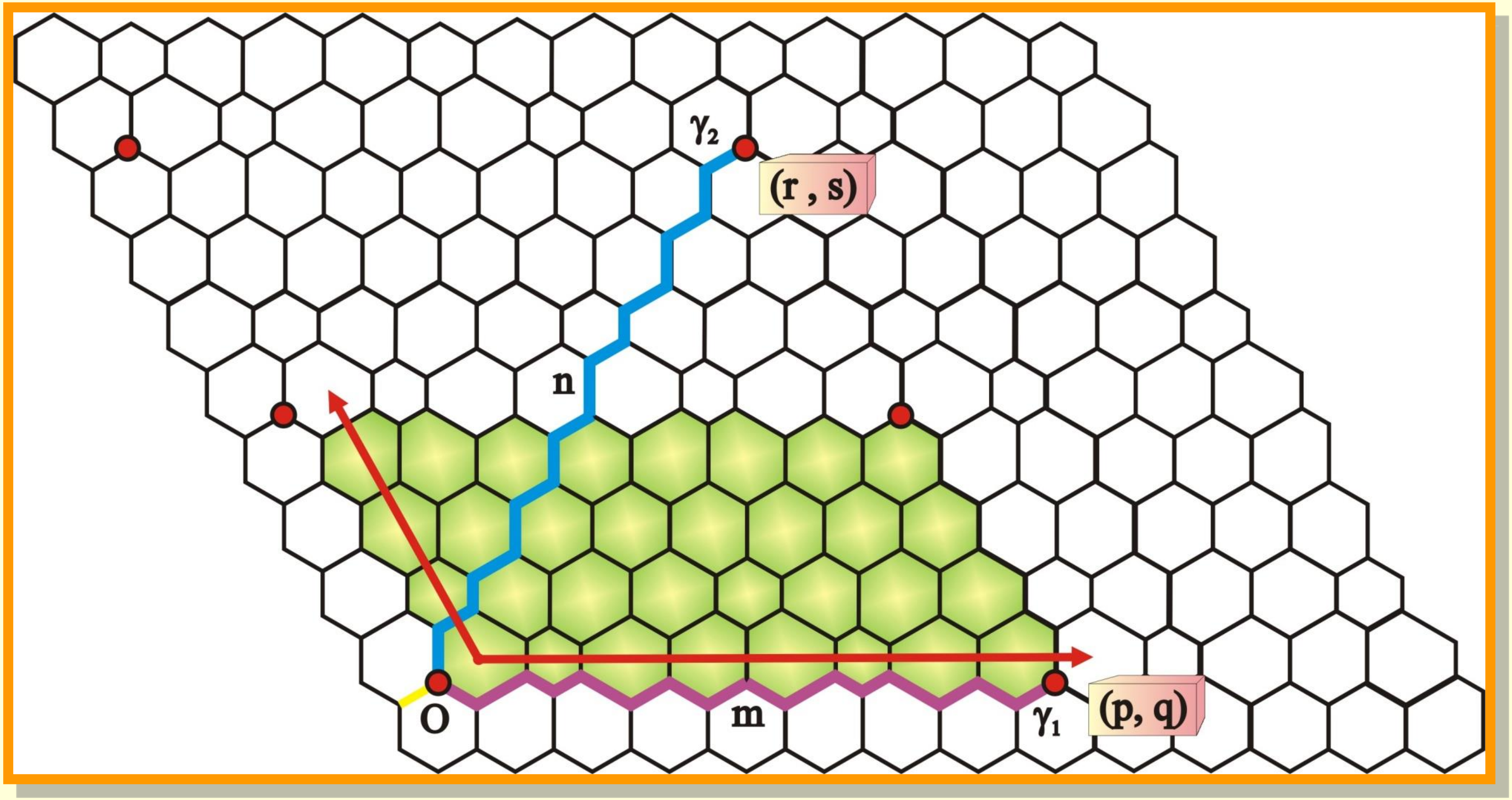
This animation gives an opportunity to imaging such gluing.



By similarity reasons, we consider only parallelograms spanned on the vectors $e = (1,0)$ and $f = (f_1,f_2)$, $f_2 > 0$. The corresponding torus will be denoted by $T^2(f)$. Now, define natural mapping from the plane to the torus



The lift of a network from torus to the plane



γ_1 and γ_2 are net geodesics forming a net basis

$M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$, $(p, q) = (r, s) = 1$, in our case $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $m = 8$, $n = 8$

Observation (A.Ivanov, I.Ptitsyna, A.Tuzhilin).
Let $d = \det M = pq - rs > 0$, then $m = u\, d$ and $n = v\, d$ for some positive integers u and v.

Thus, we can characterize our network by the triple (M, m, n).

This triple depends on the choice of the net basis (γ_1, γ_2).

The next step of encoding.

$$\left(M=\begin{pmatrix} p & r \\ q & s \end{pmatrix}, m=ud, n=vd\right) \mapsto g=\begin{pmatrix} pv & ru \\ qv & su \end{pmatrix}=\begin{pmatrix} P & R \\ Q & S \end{pmatrix}$$

Thus, we encode networks by integer matrices g with positive determinant. We call such matrices g the types of our networks. We denote the space of such types by H.

What happens with the type g if we change the net basis?

Answer: g_1 and g_2 describes the same network, if and only if they differ by J^k, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Evidently, $J^6 = E$, so each network is represented by six matrices g, gJ, gJ^2, gJ^3, gJ^4, gJ^5 (they are all different).

We denote the set of these matrices by [g], and such [g] will encode our networks.

For which types $g = \begin{pmatrix} P & R \\ Q & S \end{pmatrix}$ there exists a closed locally minimal network on a given flat torus $T^2(f)$?

Let $e = (1,0)$, $O = (0, 0)$, $A = P\,e + Q\,f$, $B = R\,e + S\,f$

The triangle $\Delta = OAB$ is called characteristic

Theorem (A.Ivanov, I.Ptitsyna, A.Tuzhilin). A closed local minimal network of the type g exists on the torus $T^2(f)$ iff all angles of the characteristic triangle are less than 120^o.

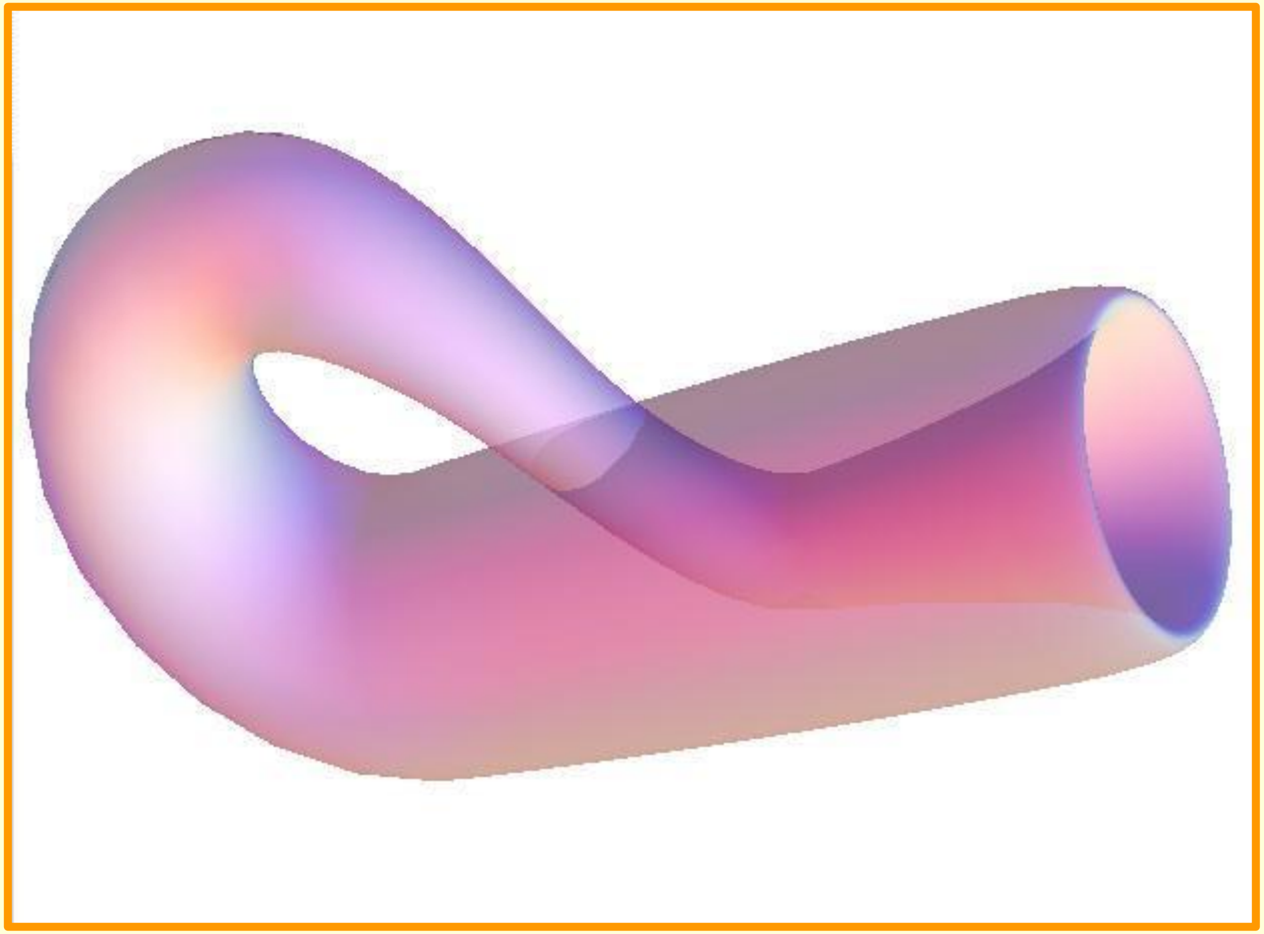
Corollary (realization of types on a tori). For any $[g]$ there exists a flat torus $T^2(f)$ and a closed LMN on it of the type g.

Corollary (infinitely many of LMN on tori). For any flat torus $T^2(f)$ there exists infinitely many closed LMN of different structure.

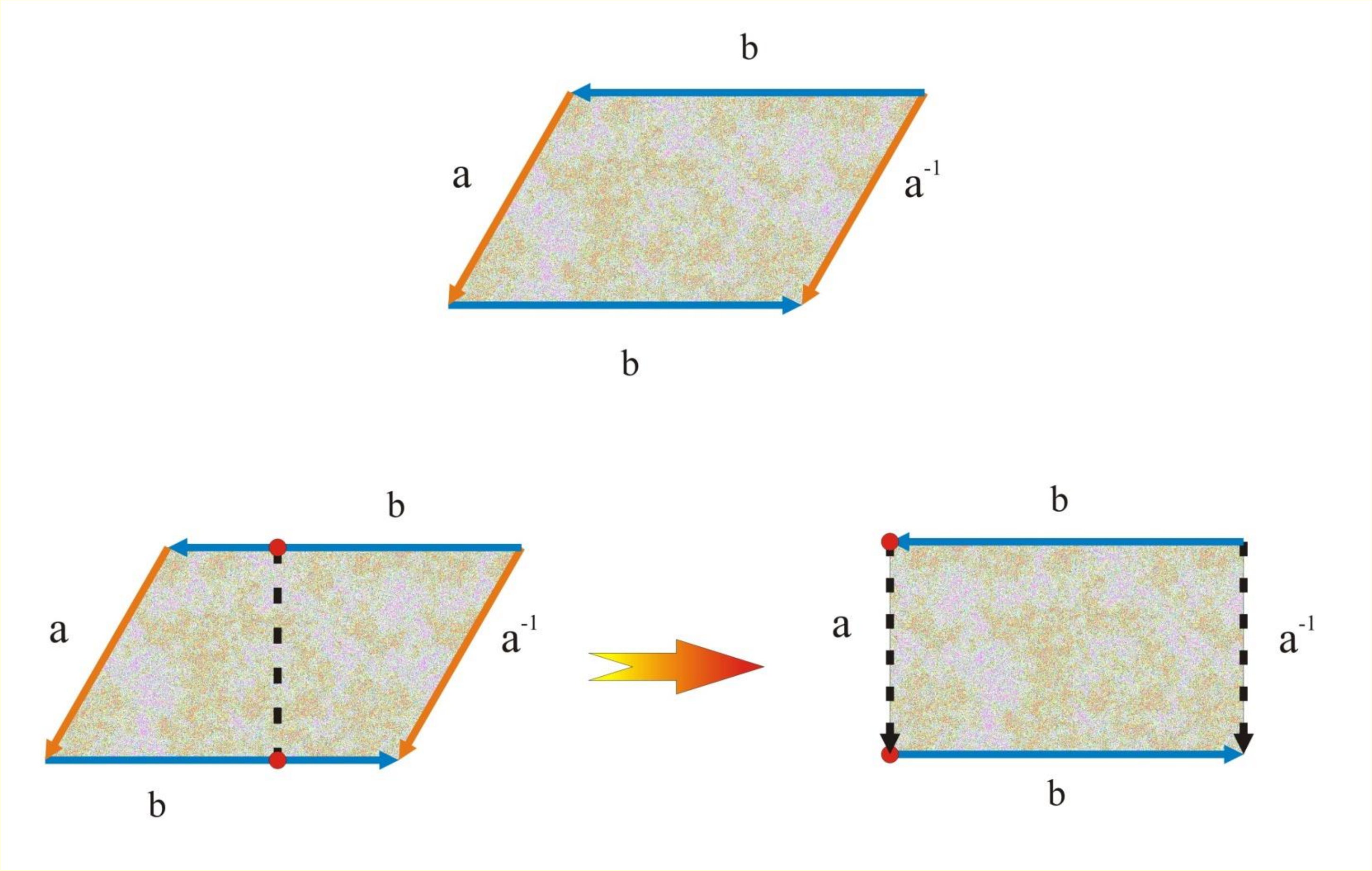
Corollary (stability). For any closed LMN Γ on a flat torus $T^2(f)$ there exists a neighborhood U of f such that for any $f' \in U$ there exists a network Γ' on $T^2(f')$ of the same type as Γ has.

(6) Reducing a new problem to a solved one.

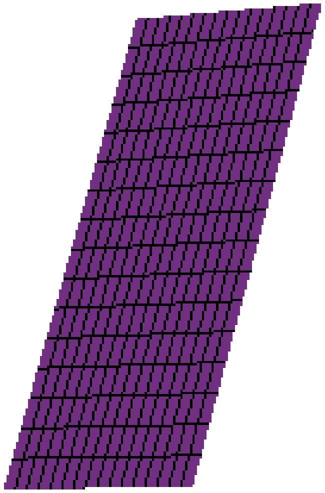
Klein bottle K^2



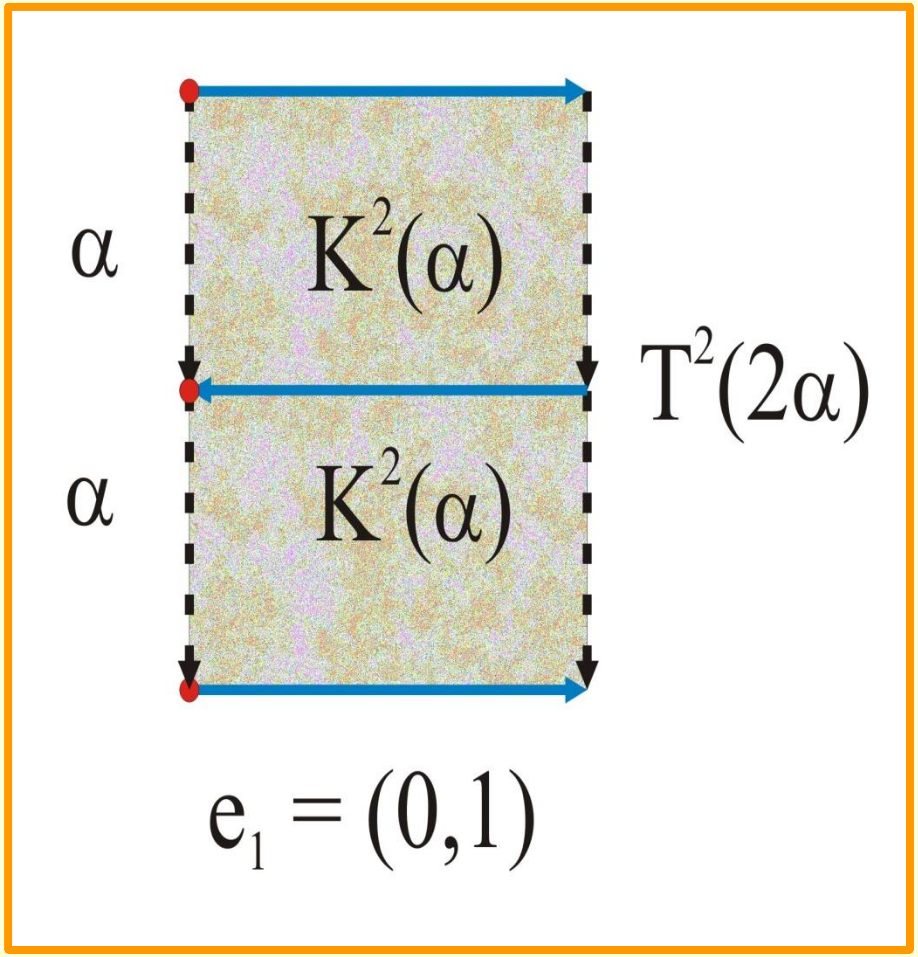
We glue Klein bottles K^2 from parallelograms. However, this can be done from rectangles.



Klein bottle K^2



Flat Klein bottle K^2 (α) and its covering by the torus $T^2(f)$, $f = (0, \alpha)$.



Covering $\nu : T^2 \rightarrow K^2$

$$\Gamma \subset K^2 \ \rightarrow \nu^{-1}(\Gamma) \subset T^2$$

Definition.

$$\mathrm{type}(\Gamma\ ) = \mathrm{type}(\nu^{-1}(\Gamma)\ )$$

Theorem (Ivanov, Ptitsina, Tuzhilin). Let Γ be a closed LMN on $K^2(\alpha)$, then there exists a net basis such that either

$$\mathrm{type}(\Gamma) = \begin{pmatrix} 2a & a \\ 0 & b \end{pmatrix}, \quad \frac{a}{b} < \sqrt{3}\,\alpha,$$

or

$$\mathrm{type}(\Gamma) = \begin{pmatrix} a & 0 \\ b & 2b \end{pmatrix}, \quad \frac{a}{b} > \frac{1}{\sqrt{3}}\,\alpha.$$

Moreover, for any such a and b there exists a closed LMN Γ on $K^2(\alpha)$ having the corresponding type.

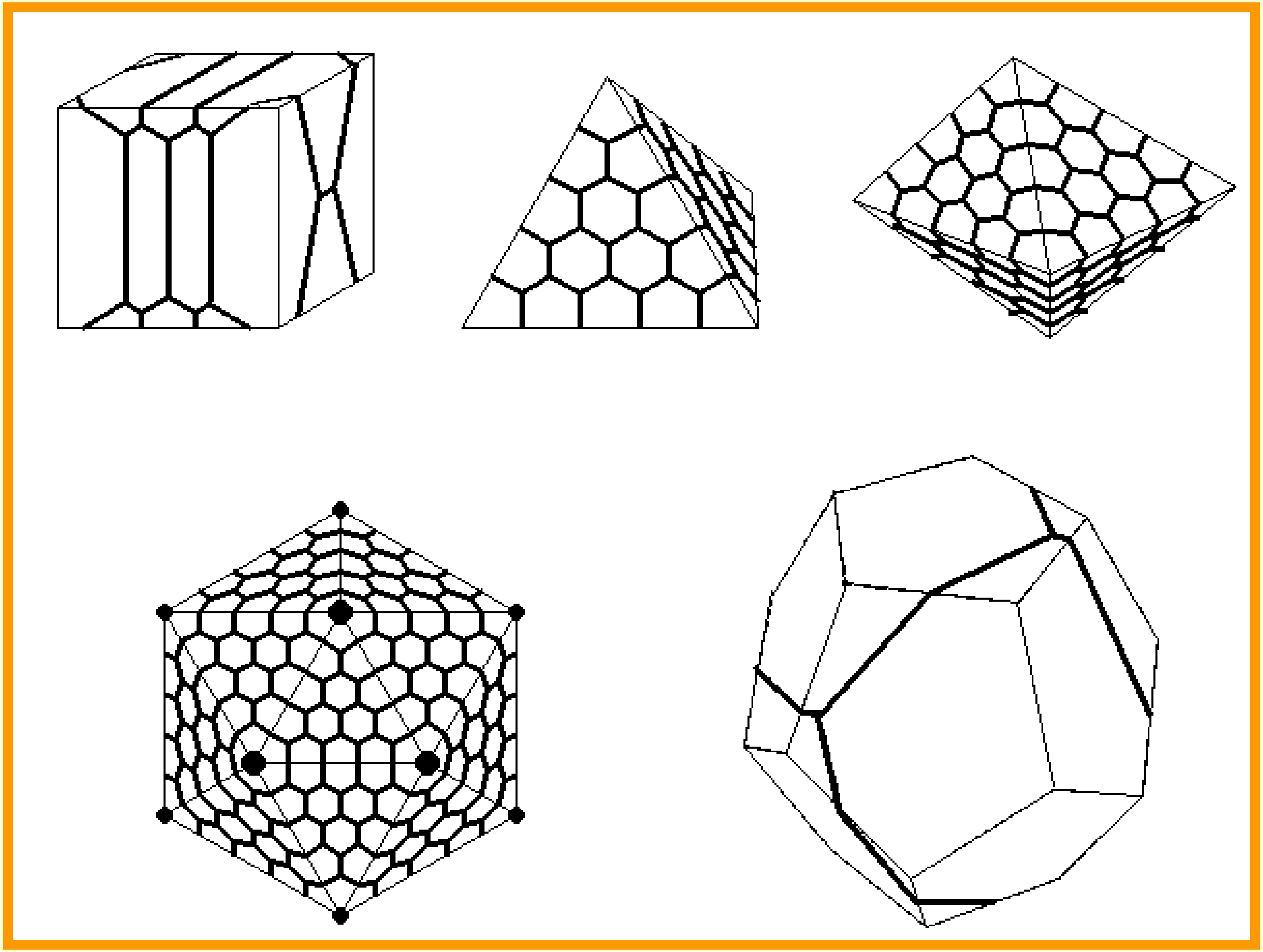
Corollaries are similar to the case of tori.

Local minimal networks on polyhedral surfaces

Consider closed networks Γ on the surface of convex polyhedron P with the vertices set Vert(P).
Thus, $\Gamma \cap \mathrm{Vert}(P) = \varnothing$.

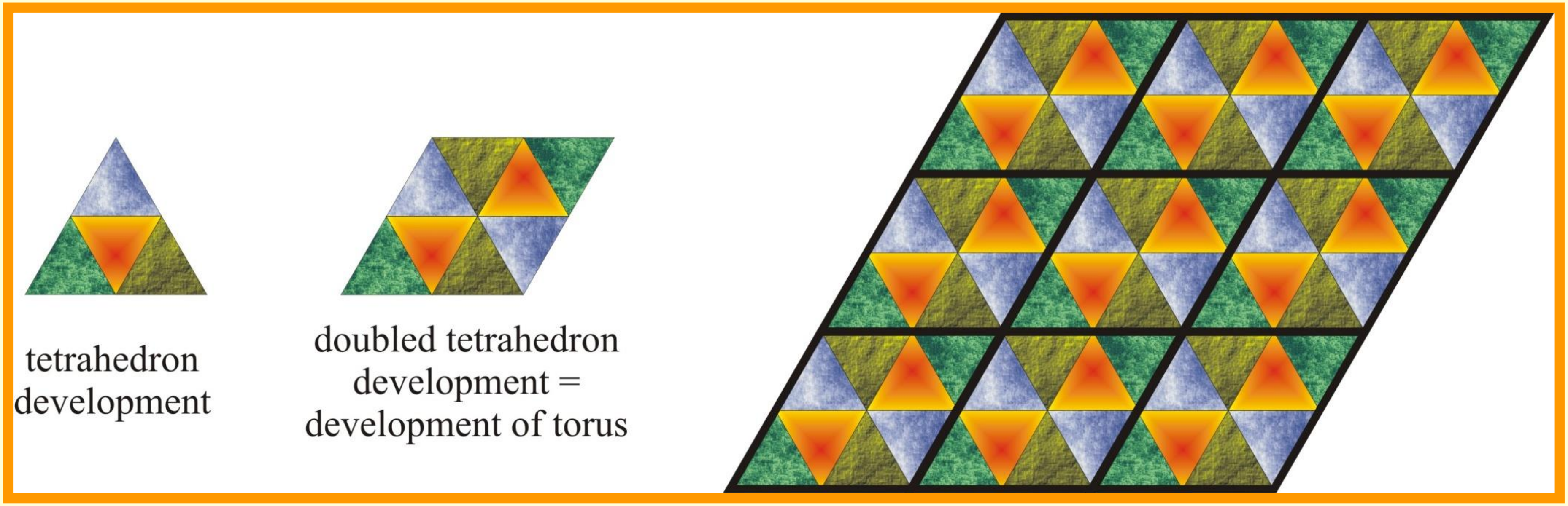
This implies that the local structure of LMN is the same as in the plane.

Examples.



Complete description of closed LMN on tetrahedron

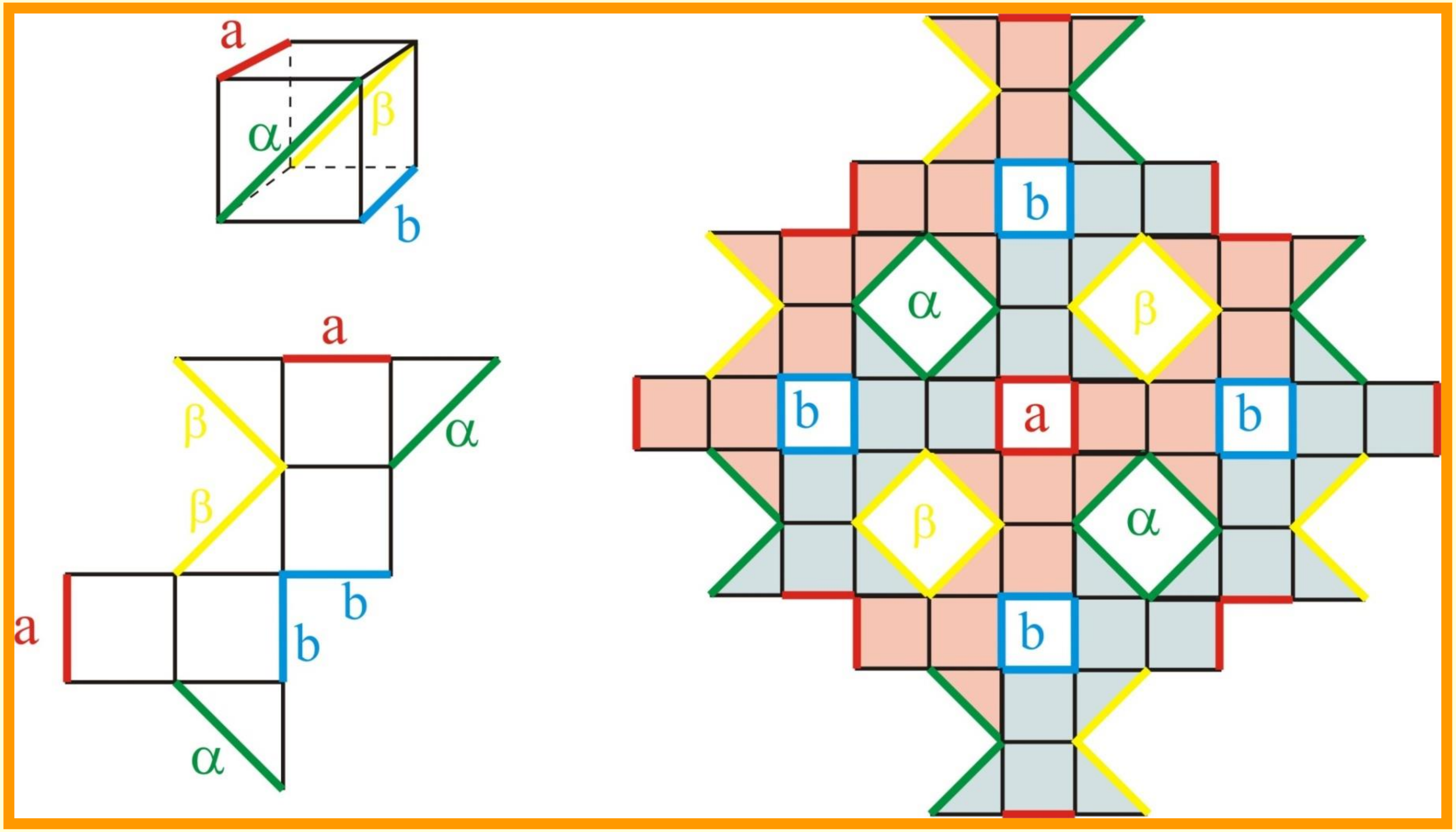
The main idea is to use branched covering $\nu : T^2 \to P$



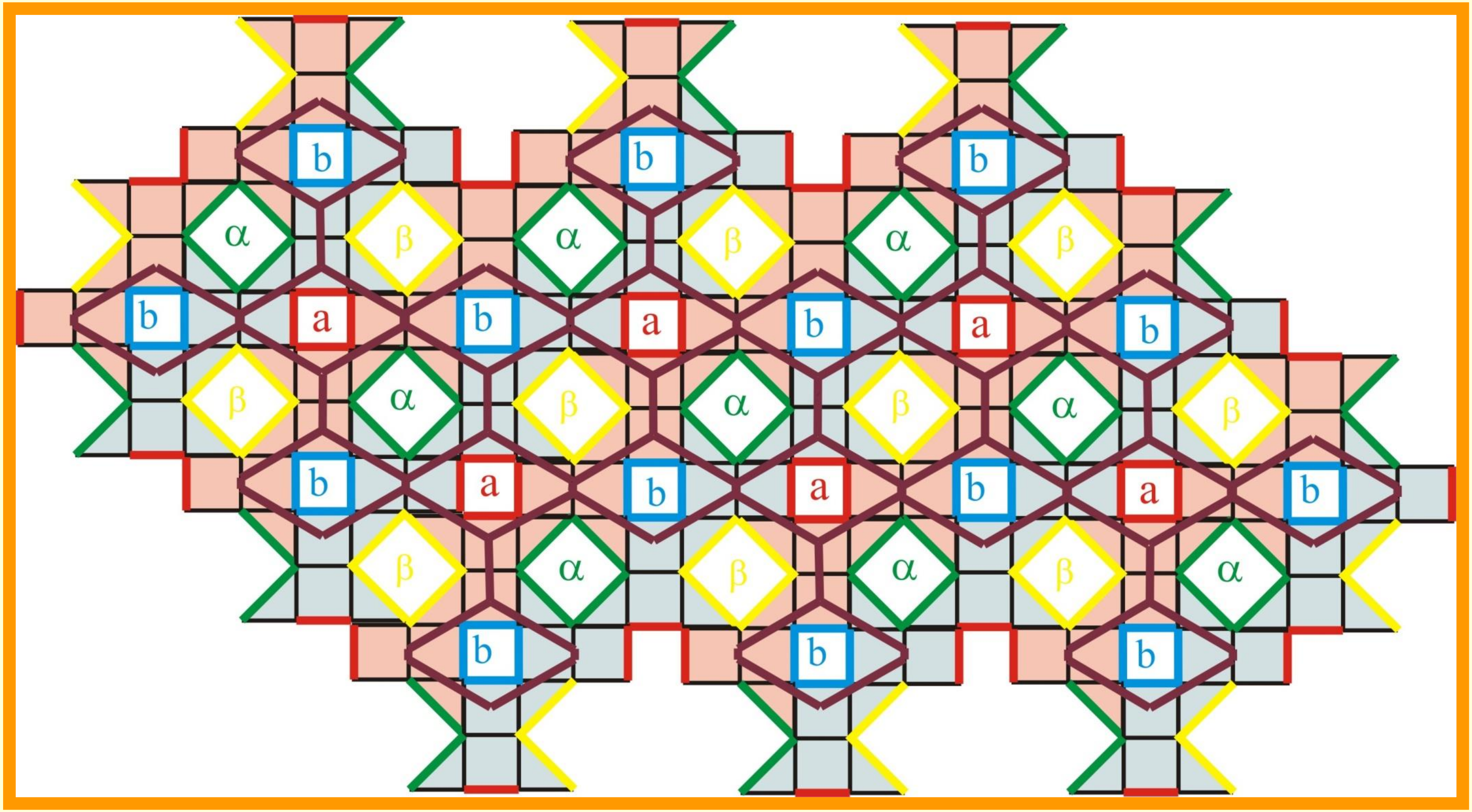
Theorems and **Corollaries** are similar to the case of tori.

Cube (the idea of D.Ablyaev, I. Ptitsina)

Construct invariant partition of $\mathbb{R}^2$ into cube's developments and holes; the union of the developments can be naturally mapped on a torus with holes; the torus with holes branching covers the cube.



Development, holes, and LMN



(7) Heuristic solutions and estimation of their accuracy.

Minimal Spanning Trees and Shortest Trees.

(X, ρ) is a metric space, $M \subset X$ is finite, $G = (V, E)$ is a graph

If $V = M$, then we say that G spans M

If $M \subset V \subset X$, then we say that G joins M

$mst(M) = \inf \{\rho(G) \mid G$ is a tree spanning $M\}$ is called the minimal length of spanning trees for M.

$smt(M) = \inf \{\rho(G) \mid G$ is a tree joining $M\}$ is called the minimal length of joining trees for M.

G is a tree spanning M, and $\rho(G) = mst(M)$, then G is called a Minimal Spanning Tree (MST) for M.

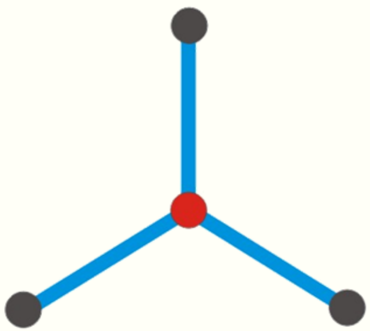
G is a tree joining M, and $\rho(G) = smt(M)$, then G is called a Shortest Tree or Steiner Minimal Tree (SMT) for M.

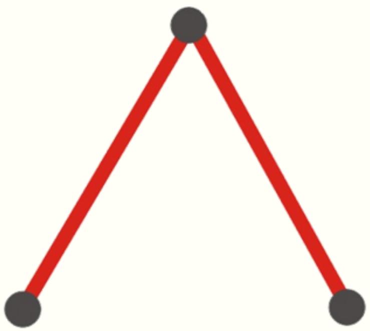
Steiner Ratio.

$sr(M) = smt(M) / mst(M)$ is called the Steiner Ratio for M (it measures the precision of MST-approximation)

$sr(X, \rho) = sr(X) = \inf \{sr(M) \mid M \subset X, M$ is finite$\}$ is the Steiner Ratio for (X, ρ) (it measures the worst precision over all MST-approximations of finite SMTs)

Example. Let M be a regular triangle in $\mathbb{R}^2$, whose sides are of the length 1, then





$\mathrm{SMT}(M) = \sqrt{3}$

$\mathrm{MST}(M) = 2$

Conjecture (Gilbert-Pollak)

$$\mathrm{sr}(M) = \frac{\sqrt{3}}{2}$$

$$\mathrm{sr}(R^2) = \frac{\sqrt{3}}{2}$$

In 1990 **D.Z.Du** and **F.K.Hwang** (Bell Labs., USA) announced a proof of Gilbert-Pollak Conjecture. However, it turns out that their proof has serious gaps.

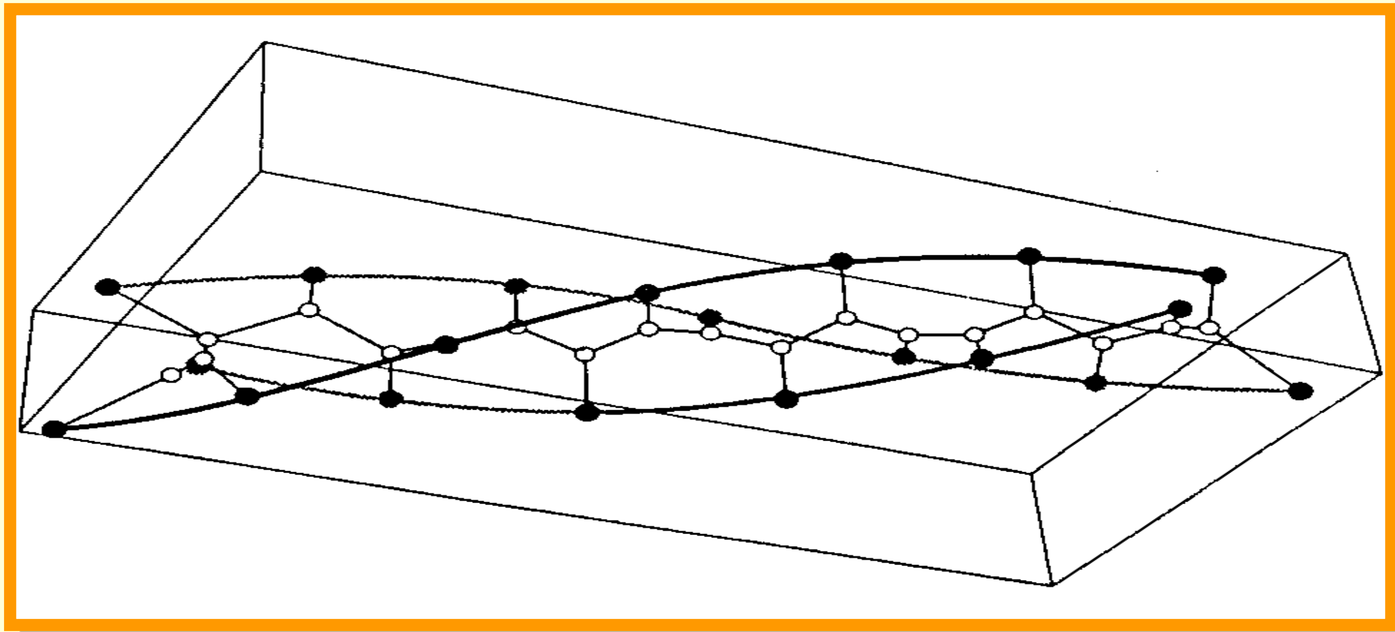
Steiner Ratio of Euclidean $\mathbb{R}^n$.

If Gilbert-Pollak conjecture is true, then the Steiner Ratio of $\mathbb{R}^2$ is attained on vertices of regular triangle.

However, for any $n \geq 3$, if $M \subset \mathbb{R}^n$ is the vertices set of a regular simplex, then $sr(M) > sr(\mathbb{R}^n)$.

Also, the best known estimation of the Steiner Ratio for $\mathbb{R}^3$ is attained at infinite set, namely,

Conjecture (W.D.Smith & J.M.Smith). The Steiner ratio for $\mathbb{R}^3$ is attained at the "sausage" infinite points boundary:



If so, the Steiner ratio of $\mathbb{R}^3$ equals

$$\sqrt{\frac{283}{700} - \frac{3\sqrt{21}}{700} + \frac{9\sqrt{11-\sqrt{21}}\,\sqrt{2}}{140}} = 0.78419\ldots$$

One more motivation.

D.Z.Du and **W.D.Smith** (1996) “proved” that if the Steiner ratio is attained on a finite subset $M \subset \mathbb{R}^n$, then the number of points in M can not be less than the value of a rapidly increasing function $f(n)$.

(Not long ago **Z.Ovsyannikov** and **B.Bednov** found a gap in their proof).

Anyway, if it’s true, then, for example,

$f(50) = 53$, $f(200) = 3\,481\,911$, etc.

This also motivates the interest to generalize SMT theory to infinite boundary sets.

(8) Extending the space of feasible systems by abstract objects to “symmetrize” the space and, as consequence, obtain simpler formulation of the laws.

Fine sets.

Definition. A set M of a metric space X is called fine if it can be spanned by a finite length tree.

Remark. Any fine set is at most countable.

Fine sets in $\mathbb{R}$.

Definition. Outer Jordan measure $\mu(M)$ of a set $M \subset \mathbb{R}$ is

$$\mu(M) = \inf \left\{ \sum_{k=1}^{N} (b_k - a_k) \mid M \subset \bigcup_{k=1}^{N} (a_k, b_k) \right\}.$$

Observation. Let $M \subset \mathbb{R}$ be bounded and countable. Then

(M is fine) $\Leftrightarrow$ ($\mu(M) = 0$).

Moreover, for a fine set $M \subset \mathbb{R}$ we have $\mathrm{mst}(M) = \mathrm{diam}(M)$.

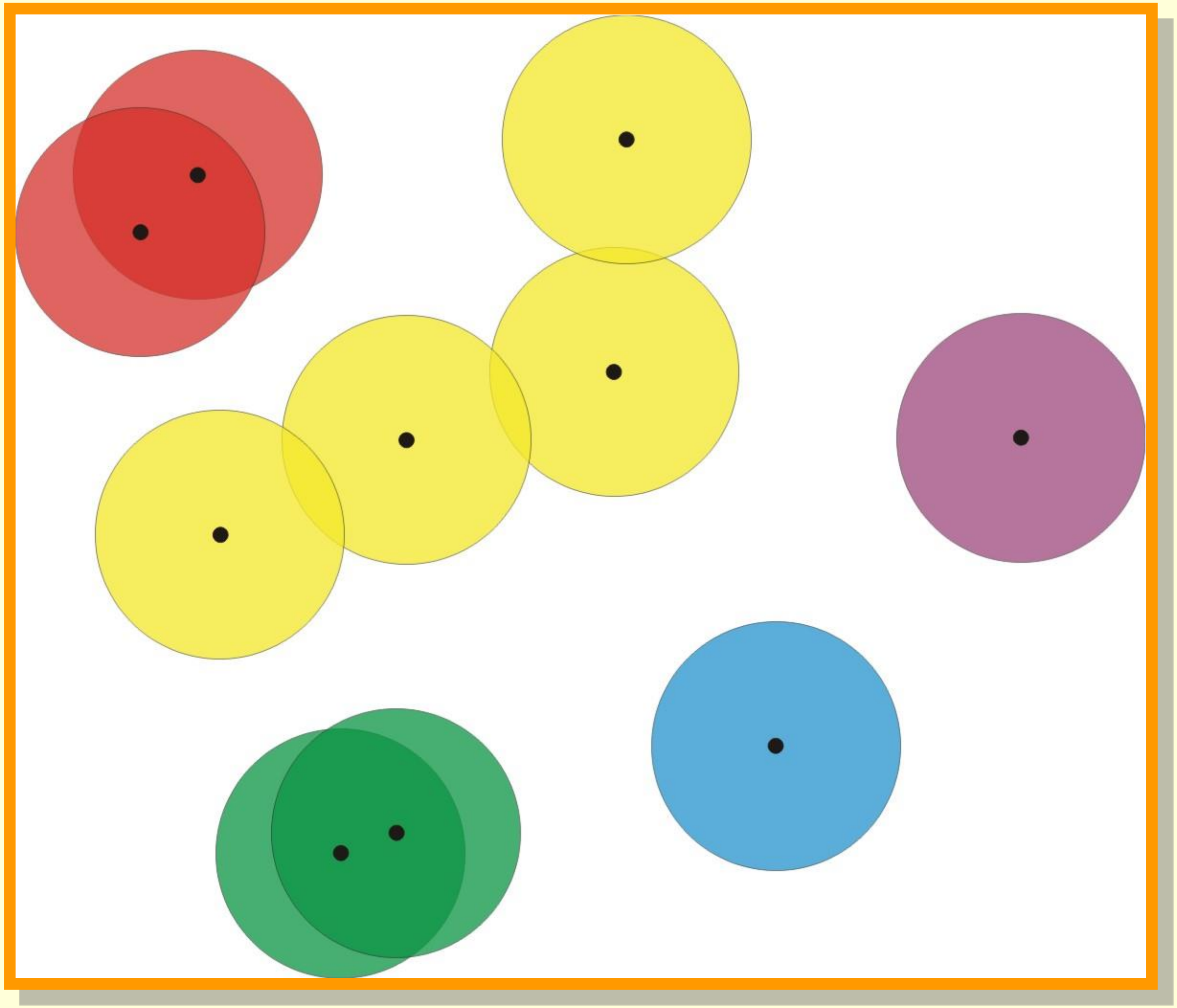
Fine sets criterion.

Let (X, ρ) be a metric space, $M \subset X$, $\lambda \geq 0$.

Let $U(\lambda, M)$ be the open λ-neighborhood of the set M, i.e., $U(\lambda, M)$ is the union of all open balls of radius λ centered at points from M.

Let $\{U_\alpha(\lambda, M)\}$ be the family of connected components of $U(\lambda, M)$, and $M_\alpha = M \cap U_\alpha(\lambda, M)$.

So $P_\lambda(M) = \{M_\alpha\}$ is a partition of M.



Observation. We have

$$P_0(M) = \{\{m\} \mid m \in M\},\ P_\infty(M) = \{M\}.$$

If $0 \le a \le b$, then $P_a(M)$ is a subpartition of $P_b(M)$.

For any subset M of X and any $\lambda \geq 0$ we put

$$\mathrm{Diam}_\lambda(M) = \Sigma\ \mathrm{diam}(c) \text{ over all } c \in P_\lambda(M).$$

Example. Let $M = \{1, 1/2, 1/3, 1/4, \ldots\}$ and $1/12 < \lambda \leq 1/6$.
Then $P_\lambda(M) = \{\{1\}, \{1/2\}, M \setminus \{1, 1/2\}\}$, thus

$$\mathrm{Diam}_\lambda(M) = 1/2.$$

Main Theorem (A.Ivanov, I.Nikonov, A.Tuzhilin).

Let M be a bounded countable subset of a metric space, and we put $\pi_\lambda(M) = \# P_\lambda(M)$. Then M is fine iff

(1) $\int_0^{\mathrm{diam}(M)} \pi_\lambda(M)\, d\lambda < \infty$, and

(2) $\mathrm{Diam}_\lambda(M) \to 0$ as $\lambda \to 0$.

Moreover, for a fine set M we can calculate the length $\mathrm{mst}(M)$ of Minimal Spanning Tree on M as follows:

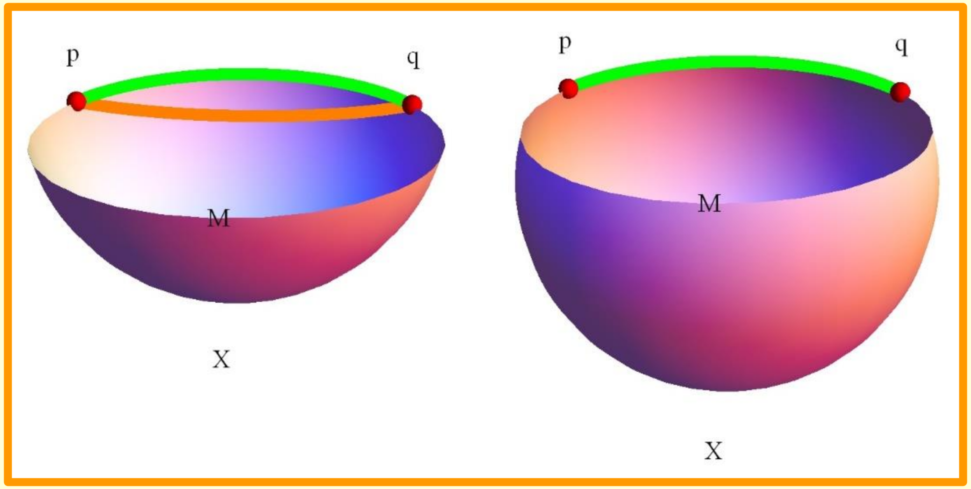
$$\mathrm{mst}(M) = \int_0^{\mathrm{diam}(M)} \pi_\lambda(M)\, d\lambda - \mathrm{diam}(M).$$

(9) Generalizations.

M. Gromov Minimal Fillings Problem.

n-dimensional manifold X with a metric d is called a filling of (n-1)-dimensional manifold M with a metric ρ,

if $M = \partial X$, and $\rho(p,q) \le d(p,q)$ for all $p, q \in M$.



Problem (M.Gromov). Given $\mathcal{M} = (M, \rho)$, find the least possible volume $mf(\mathcal{M})$ of fillings $\mathcal{X} = (X, d)$ for $\mathcal{M}$, and describe the fillings $\mathcal{X}$ for which $mf(\mathcal{M}) = volume(\mathcal{X})$.

We discuss one-dimensional stratified variant of the problem:

$\mathcal{M} = (M, \rho)$ is a finite pseudometric space,

$\mathcal{X} = (X, d)$ is generated by a weighted graph.

Formalization:

$G = (V, E)$ is a connected graph joining M, i.e., $M \subset V$

$\omega: E \to \mathbb{R}_+$ is called a weight function on G

$\mathcal{G} = (G, \omega) = (V, E, \omega)$ is called a weighted graph

for each subgraph $\mathcal{H} \subset \mathcal{G}$ its weight is $\omega(\mathcal{H}) = \sum_{e \in E(\mathcal{H})} \omega(e)$

the distance on V generated by ω is

$d_\omega(x,y) = \min_\gamma \{\omega(\gamma) \mid \gamma$ is a path in G joining $x, y \in V\}$.

$\mathcal{M} = (M, \rho)$, where ρ is a distance on M

$\mathcal{G} = (V, E, \omega)$ is a connected weighted graph joining $\mathcal{M}$

$\mathcal{G}$ is called a filling of $\mathcal{M}$, iff

$$\rho(x,y) \leq d_\omega(x,y) \text{ for all } x, y \in M$$

each $x \in M \subset V$ is called a boundary vertex

each $x \in V \setminus M$ is called an interior vertex

$\mathcal{M}$ is called the boundary of the filing $\mathcal{G}$

The number $\inf \omega(\mathcal{G})$ over all fillings $\mathcal{G}$ of $\mathcal{M}$ is called the minimal weight of fillings and is denoted by $mf(\mathcal{M})$

A filling $\mathcal{G}$ of $\mathcal{M}$ such that $\omega(\mathcal{G}) = mf(\mathcal{M})$ is called minimal

Remark.
Our formalization is slightly differ from the one in the Gromov Problem: in fact, we join 0-dimensional manifold $\mathcal{M}$ with 0-dimensional manifold (V, d_ω), instead of to do that with some 1-dimensional manifold.

To make our the formalization more close, we can consider $\mathcal{G}$ as a 1-dimensional stratified manifold which is glued from segments of lengths given by the weight function ω. Now, may be, it’s more clear why we call $\mathcal{M}$ by the boundary of $\mathcal{G}$.

Existence.

A weighted graph is called nondegenerate if its weight function is strictly positive.

Proposition. Let $\mathcal{M}$ be a finite pseudometric space. Then

1) there exists a minimal filling for $\mathcal{M}$;
2) among all minimal fillings of $\mathcal{M}$ there exists a binary tree for which $\mathcal{M}$ consists of all its vertices of degree 1;
3) if $\mathcal{M}$ is a metric space, then there exists a nondegenerate minimal filling $\mathcal{G}$ for $\mathcal{M}$, such that $\mathcal{M}$ contains all vertices of $\mathcal{G}$ having degree 1 and 2. Here the graph $\mathcal{G}$ is, obviously, a tree.

Important agreements.

(1) In what follows, $\mathcal{G}$ is always a tree.

(2) For binary trees, the phrases "$\mathcal{G}$ joins $\mathcal{M}$" means, in addition, that $\mathcal{M}$ coincides with the set of all vertices of $\mathcal{G}$ having degree 1.

(3) For arbitrary trees, the phrases "$\mathcal{G}$ joins $\mathcal{M}$" means, in addition, that $\mathcal{M}$ contains all vertices of $\mathcal{G}$ having degree 1 and 2.

Additive metric spaces, minimal fillings, and uniqueness.

A finite metric space (M, ρ) is called additive, if it has a filling (G, ω) such that $\rho = d_\omega|_M$. Such weighted tree (G, ω) is called a generator of (M, ρ).

Remark. The criterion of additivity called four points condition is well-known. Also, it's known that nondegenerate generator is unique and there exists an effective algorithm for its construction.

Proposition. The generator of an additive metric space is its minimal filling. Vise versa, each minimal filling of an additive metric space is its generator.

Thus, for additive spaces their nondegenerate minimal fillings are unique, and one can construct the corresponding minimal fillings by means of well-known effective algorithm.

Remark. For general metric spaces one can construct all minimal fillings by means of linear programming applied to each possible binary topology of the filling (notice that the number of such topologies grows exponentially on the cardinality of the metric space).

More precise, consider a binary tree $\mathcal{G} = (G, \omega)$ such that its weights $\omega_i \geq 0$ are unknown and it joins $\mathcal{M}$. Then the equations guarantees that $\mathcal{G}$ is a filling of $\mathcal{M}$ are linear inequalities.

On the other hand, $\omega(\mathcal{G}) = \sum_i \omega_i$, and, to find a minimal filling, we need first to minimize $\omega(\mathcal{G})$ over all admissible ω (it is the standard linear programming problem), and than minimize the obtained values over all binary trees G.

Conjecture. One-dimensional minimal filing construction problem is NP-hard.

Remark. There exist metric spaces for which there is no uniqueness (for example, for vertices of a regular n-gon in the Euclidean plane, $n > 3$).

However, in some important case uniqueness occurs.

Metric space is called rigid, if its nondegenerate minimal filling is unique.

Corollary. Each additive space is rigid.

"Generic" results

All metric spaces consisting of n points can be naturally identified with a convex cone in $\mathbb{R}^{n(n-1)/2}$. We say, that some property holds for a generic metric space, if for any n this property is valid for an everywhere dense set of n-point metric spaces.

Is it true that a generic metric space is rigid?
The answer is false (**Z. Ovsyannikov**).

The following result is due to **A. Eremin.**

Proposition. Each generic finite metric space has a minimal filling which is a nondegenerate binary tree.

Description of all minimal fillings.

Proposition.
(M, ρ) is a finite pseudometric space
(G, ω) is a minimal filling for (M, ρ)
then (G, ω) is a minimal filling for $(M, d_\omega|_M)$

Corollary. The set of all possible minimal fillings (recall that we consider only trees) of all finite pseudometric spaces coincides with the set of all weighted minimal trees.

Minimal Fillings as Shortest Trees.

Let $\mathbb{R}^n_\infty$ denote $\mathbb{R}^n$ with the norm

$$\|(v^1,\ldots,v^n)\|_\infty = \max\{|v^1|,\ldots,|v^n|\}.$$

By ρ_∞ we denote the corresponding metric.

For a metric space $\mathcal{M} = (M, \rho)$ such that $M = \{p_1,\ldots, p_n\}$ we put $\rho_{ij} = \rho(p_i,p_j)$ and construct $\varphi_{\mathcal{M}} : M \to \mathbb{R}^n_\infty$ as follows:

$$\varphi_{\mathcal{M}}: \; p_i \mapsto (\rho_{i1},\ldots, \rho_{in}).$$

Proposition. The map $\varphi_{\mathcal{M}}$ is an isometric embedding.

$\mathcal{G} = (V, E, \omega)$ is a filling of $\mathcal{M} = (M, \rho)$

Define

$$E' = E \cup \{p_i p_j\}_{i \neq j} \ ,$$

$$\omega' : E' \to \mathbb{R}_+$$

$$\omega'|_{E \setminus E'} = \omega, \ \omega'(p_i p_j) = \rho_{ij}$$

Then $\mathcal{G}' = (V, E', \omega')$ is a filling of $\mathcal{M}$

Define $\Gamma_{\mathcal{G}} : V \to \mathbb{R}^n_\infty$

$$\Gamma_{\mathcal{G}} : v \mapsto (d_{\omega'}(v, p_1), \ldots, d_{\omega'}(v, p_n)).$$

Proposition.

Let $\mathcal{G} = (G, \omega)$ be a minimal filling for $\mathcal{M} = (M, \rho)$. Then

1) $\Gamma_{\mathcal{G}}|_M = \varphi_{\mathcal{M}}$
2) for any $xy \in E$ we have $\rho_\infty(\Gamma_{\mathcal{G}}(x), \Gamma_{\mathcal{G}}(y)) = \omega(xy)$
3) $\Gamma_{\mathcal{G}}$ is a shortest tree with the boundary $\varphi_{\mathcal{M}}(M)$
4) each shortest tree joining $\varphi_{\mathcal{M}}(M)$ is a minimal filling for $(\varphi_{\mathcal{M}}(M), \rho_\infty) \approx (M, \rho)$.

Exact paths.

$\mathcal{G} = (V, E, \omega)$ is a filling of $\mathcal{M} = (M, \rho)$

A path γ in $\mathcal{G}$ joining $p, q \in M$ is called a boundary one.

A boundary path is called irreducible if it does not contain another boundary path.

A boundary path is called exact if $\omega(\gamma) = \rho(p, q)$.

Proposition.

(1) A metric space is additive, iff all boundary paths in any its minimal filling are exact.

(2) In any minimal filling each boundary path contained in an exact path is exact itself.

(3) In any minimal filling each boundary path consisting of at most 2 edges is exact.

Corollary. If a minimal filling $\mathcal{G}$ of a metric space $\mathcal{M}$ is star-like, i.e., $\mathcal{G}$ has a unique non-boundary vertex and this vertex is joined with all boundary ones, then $\mathcal{M}$ is additive, and, thus, $\mathcal{G}$ is its generator.

Given a filling (V, E, ω), a set $F \subset E$ is called exact if it belongs to an exact path.

Proposition. For any minimal filling, every path consisting of at most 2 edges is exact.

Tours and Perimeters.

$\mathcal{M} = (M, \rho)$ is a finite metric space.

Enumerate the points of M in an arbitrary way:
$M = \{p_1, \ldots, p_n\}$ and put $p_{i+n} = p_i$ for any integer i.
We call the obtained cyclic order by a tour of M.

The perimeter of $\mathcal{M}$ w.r.t. the tour π is the value

$$P_\pi = \sum_{i=1,\ldots,n} \rho\,(p_i, p_{i+1}).$$

Also, we put $p_\pi = P_\pi / 2$ and call it the half-perimeter of $\mathcal{M}$ w.r.t. the tour π.

The half-perimeter $p(\mathcal{M})$ of $\mathcal{M}$ is the smallest value p_π over all tours π.

Proposition (O.Rubleva). For any metric space $\mathcal{M}$ we have

$$p(\mathcal{M}) \leq mf(\mathcal{M}).$$

Moreover, the equality holds iff $\mathcal{M}$ is additive.

Remark. The previous proposition is a new additivity criterion.

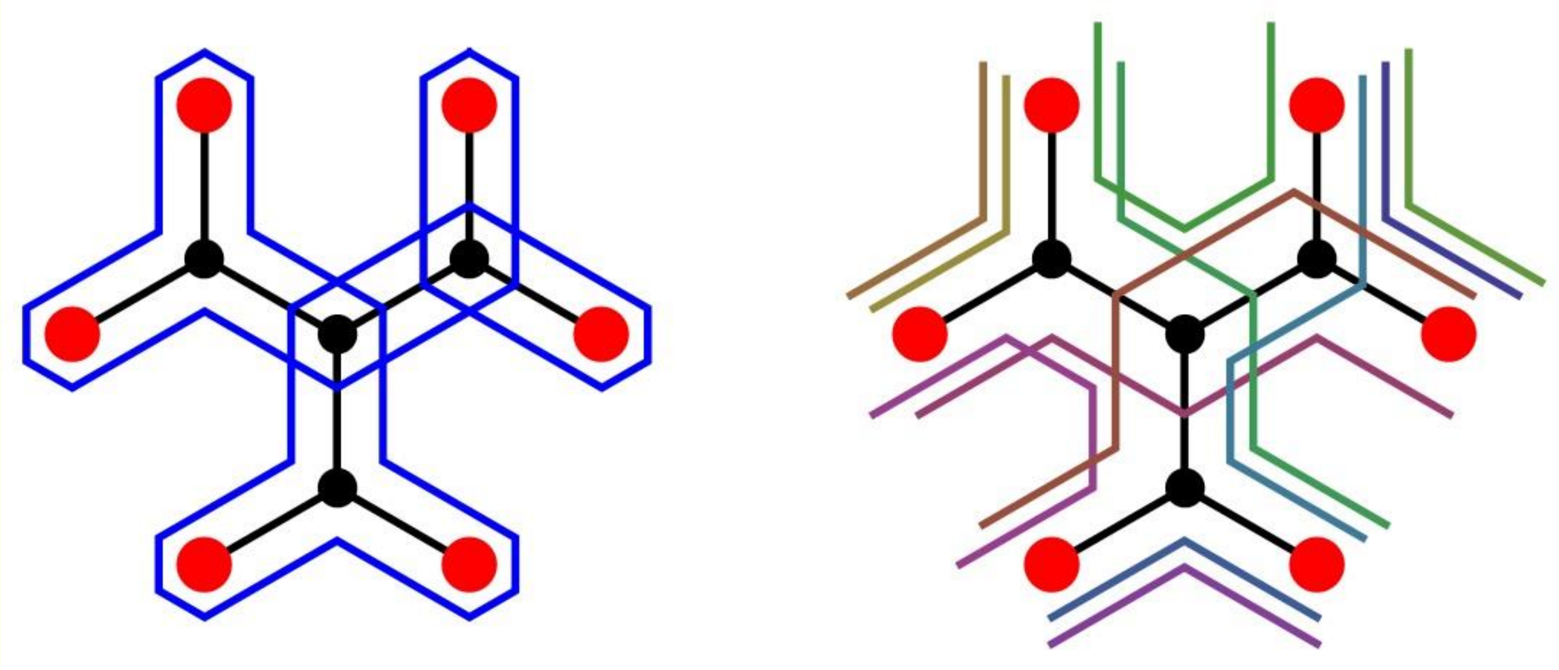
Multitours.

An Euler tour in a (connected) graph is a cycle containing all edges of the graph and passing each of them just one time.

$\mathcal{M} = (M, \rho)$ is a pseudometric space, G is a tree joining M

G^{2k} is graph obtained from G by taken each edge $2k$ times

G^{2k} possesses an Euler cycle consisting of irreducible in G boundary paths



This Euler cycle generates a bijection $\pi : \coprod_{i=1}^{k} M \to \coprod_{i=1}^{k} M$,
which is called multitour.

We put $p(\mathcal{M}, G, \pi) = \frac{1}{2k} \sum_{v \in M} \rho(v, \pi(v))$

The set of all multitours for $\mathcal{M}$ and G is denoted by $O_\mu(\mathcal{M},G)$

Formula for the weight of minimal filling.

Theorem (A. Eremin) Let $\mathcal{M} = (M, \rho)$ be a finite pseudometric space. Then

$$\mathrm{mf}(\mathcal{M}) = \min_G \max_\pi \{ p_\pi(\mathcal{M}, G, \pi) \mid \text{over all } \pi \in O_\mu(\mathcal{M}, G) \text{ and binary trees } G \text{ joining } \mathcal{M}\}.$$

Remark. The idea of such kind formula belongs to **A.Ivanov** and **A.Tuzhilin**, who stated the conjecture that the previous formula holds for tours ($k = 1$).
A.Eremin and **Z.Ovsyannikov** constructed a counter-example.
Than **A.Eremin** proved, that if one changes the tours to multitours, the formula will become true.

Generalized minimal fillings.

Expand the class of weighted trees by permitting any weights of the edges (not only nonnegative).

The corresponding objects are called by
generalized weighted graphs,
generalized fillings, and
minimal generalized fillings.

The weight of minimal generalized filling $\mathcal{G}$ of a pseudometric space $\mathcal{M}$ is denoted by $\mathrm{mf}\_(\mathcal{M})$.

Theorem (A.Ivanov, Z.Ovsyannikov, N.Strelkova, A.Tuzhilin) Among minimal generalized fillings of an arbitrary finite pseudometric space, there exists a filling with nonnegative weight function, i.e., there exists a minimal filling. Therefore, $mf\_ = mf$.

Remark. This theorem is one of key-points in the proof of Eremin’s formula on the weight of minimal filling.

Generalized additive spaces.

A.Ivanov and **A.Tuzhilin** raised the following question:
Is it true that if all the tours w.r.t. a given tree joining a pseudometric space have the same weight, then the space is additive?
The answer turns to be negative (**Z.Ovsyannikov**).

A finite pseudometric space $\mathcal{M} = (M, \rho)$ is called pseudo-additive, if $\rho = d_\omega$ for a generalized weighted tree (G, ω) joining $\mathcal{M}$.

Theorem (Z. Ovsyannikov)
Let $\mathcal{M} = (M, \rho)$ be a finite metric space.
Then the following statements are equivalent.

- There exist a tree G such that M coincides with the set of degree 1 vertices of G, and all the half-perimeters $p(\mathcal{M}, G, \pi)$ of M corresponding to the tours around G are equal to each other.

- The space $\mathcal{M}$ is pseudo-additive with the generating tree (G, ω_0) for ω_0 minimizing $\omega(G)$ over all generalized fillings of the form (G, ω).

Ratios.

For any subset M of a metric space $\mathcal{X} = (X, \rho)$ we have 3 numbers: $\mathrm{mst}(M) \geq \mathrm{smt}(M) \geq \mathrm{mf}(M)$.

We have already defined

the Steiner ratio $\mathrm{sr}(M) = \mathrm{smt}(M)/\mathrm{mst}(M)$;

Now we define two more ratios:

the Steiner-Gromov ratio $\mathrm{sgr}(M) = \mathrm{mf}(\mathcal{M})/\mathrm{mst}(M)$;

the Steiner subratio $\mathrm{ssr}(M) = \mathrm{mf}(\mathcal{M})/\mathrm{smt}(M)$.

The greatest lower bounds of these ratios over all finite $M \subset X$ are called the ones of $\mathcal{X}$ and are denoted by $\mathrm{sr}(\mathcal{X})$, $\mathrm{sgr}(\mathcal{X})$, $\mathrm{ssr}(\mathcal{X})$, resp.

If we consider the infinums over all $M \subset X$ consisting of at most n points, then we obtain n-points ratios of $\mathcal{X}$ and denote them by $sr_n(\mathcal{X})$, $sgr_n(\mathcal{X})$, $ssr_n(\mathcal{X})$ resp.

Remark. The Steiner ratio is classical. The other two ratios where introduced recently by **A.Ivanov** and **A.Tuzhilin**.

Remark. The Steiner ratio is very difficult to calculate, and its values are known for a few metric spaces only. The other ratios seem simpler, and, perhaps, they may be useful to get the results concerning the Steiner ratio.

Steiner–Gromov ratio

$\mathrm{sgr}(M) = \mathrm{mf}(\mathcal{M})/\mathrm{mst}(M)$.

Theorem (**A.Pakhomova**).
For any metric space $\mathcal{X}$ the estimate $\mathrm{sgr}_n(\mathcal{X}) \geq n/(2n-2)$ holds.
This estimate is exact, i.e., for any $n \geq 3$ there exists a metric space $\mathcal{X}_n$ such that $\mathrm{sgr}_n(\mathcal{X}) = n/(2n-2)$.

For any metric space $\mathcal{X}$ the estimate $1/2 \leq \mathrm{sgr}(\mathcal{X}) \leq 1$ holds.
For any $s \in [1/2, 1]$ there exists a metric space $\mathcal{X}$ such that $\mathrm{sgr}(\mathcal{X}) = s$.

Proposition. If $\mathcal{X}$ contains a regular simplex, then

$$\mathrm{sgr}_n(\mathcal{X}) = n/(2n-2).$$

Proposition.

We have $\mathrm{sgr}(\mathcal{X}) \leq \mathrm{sg}(\mathcal{X})$.

Thus, if $\mathrm{sg}(\mathcal{X}) = 1/2$, then $\mathrm{sg}(\mathcal{X}) = 1/2$.

Corollary.

Let L^n be n-dimensional Lobachevski space, $n \geq 2$.

Then $\mathrm{sg}(L^n) = 1/2$.

Let $\mathbb{R}^n_p$ denote $\mathbb{R}^n$ with the norm $\|(v_1,\ldots,v_n)\|_p = (\sum_i |v_i|^p)^{1/p}$.

Proposition (A.Pakhomova).

Let $\mathcal{X}$ be either the space $\mathbb{R}^n_p$ for $1 \leq p \leq \infty$, or the space of words over an alphabet $A = \{a_1, \ldots , a_k\}$, $k \geq 2$, endowed with the Levenstein metric.

Then for all $n \geq 2$ the following relations hold:

$$\mathrm{sgr}_n(\mathcal{X}) = n/(2n-2), \quad \mathrm{sgr}(\mathcal{X}) = 1/2.$$

Proposition (**Z.Ovsyannikov**). The Steiner–Gromov ratio of the metric space of all compact subsets of Euclidean plane endowed with Hausdorff metric equals $1/2$.

Proposition (**V.Mishchenko**). The Steiner–Gromov ratio of an arbitrary n-dimensional Riemannian manifold is less than or equal to the Steiner–Gromov ratio of the Euclidean space $\mathbb{R}^n$.

Steiner subratio $ssr(M) = mf(\mathcal{M})/smt(M)$.

Theorem (A.Pakhomova).
For any metric space $\mathcal{X}$ the estimate $ssr_n(\mathcal{X}) \geq n/(2n-2)$ holds.
This estimate is exact, i.e., for any $n \geq 3$ there exists a metric space $\mathcal{X}_n$ such that $ssr_n(\mathcal{X}) = n/(2n-2)$.

For any metric space $\mathcal{X}$ the estimate $1/2 \leq ssr(\mathcal{X}) \leq 1$ holds.
For any $s \in [1/2, 1]$ there exists a metric space $\mathcal{X}$ such that $ssr(\mathcal{X}) = s$.

Proposition (some partial results).

1) $\mathrm{ssr}_2(X) = 1$

2) $\mathrm{ssr}_3(\mathbb{R}^n) = \sqrt{3}/2$ **(A.Ivanov, A.Tuzhilin)**

3) $\mathrm{ssr}_4(\mathbb{R}^2) = \sqrt{3}/2$ **(E.Stepanova)**

4) $\mathrm{ssr}_5(\mathbb{R}^2) < 0.8562 < \sqrt{3}/2$ **(Z.Ovsyannikov)**

5) $\mathrm{ssr}_4(\mathbb{R}^3) = (2\sqrt{3}+\sqrt{5})/7 < 0.82 < \sqrt{3}/2$ **(Z.Ovsyannikov)**

Definition (B.Bednov, P.Borodin)

$\mathrm{ssr}(d) = \inf\{\mathrm{ssr}(V) \mid V$ is a Banach space of dimension $d\}$.

Proposition (B.Bednov, P.Borodin)

$$3/4 \le \mathrm{ssr}(2) \le 5/6, \qquad \frac{2}{\sqrt{5}+1} \le ssr(3) \le \frac{4}{5}$$

Proposition (**A.Pakhomova**).
Let $\mathcal{X}$ be the space of words over an alphabet $A = \{a_1, . . . , a_k\}$, $k \geq n - 1$, endowed with the Levenstein metric.
Then for all $n \geq 2$ the following relations hold:

$$\mathrm{ssr}_n(\mathcal{X}) = n/(2n - 2).$$

Proposition (**Z.Ovsyannikov**). Let $\mathcal{C}$ be the metric space of all compact subsets of Euclidean plane endowed with the Hausdorff metric. Then

$$\mathrm{ssr}_3(\mathcal{C}) = 3/4,\ \mathrm{ssr}_4(\mathcal{C}) = 2/3,\ \mathrm{ssr}(\mathcal{C}) = 1/2.$$

Thank You!