Asymptotic behavior of the interacting cells model

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Biological interpretation

The main aim of this talk is to discuss general biological problem related to spread of cell destruction (burn, cancer, etc).

Destruction degree of surrounding cells has an influence on vitality probability of the cell.

Our purpose is to estimate what are the chances to survive for every cell in this situation and how the destruction will spread.

Basic definitions

Cellular finite tissue

$\mathrm{T} = \{1, \dots, m\} \times \{1, \dots, m\}, \mathrm{m} \in \mathbb{N}$

(model of tissue; $(\mathrm{i}, \mathrm{j}) \in T$ is called **cell**)

Standard window of observation

$\mathrm{S} = \{-r, \dots, r\} \times \{-r, \dots, r\}, \mathrm{r} \in \mathbb{N}.$

(the range of destruction influence)

Weight coefficients

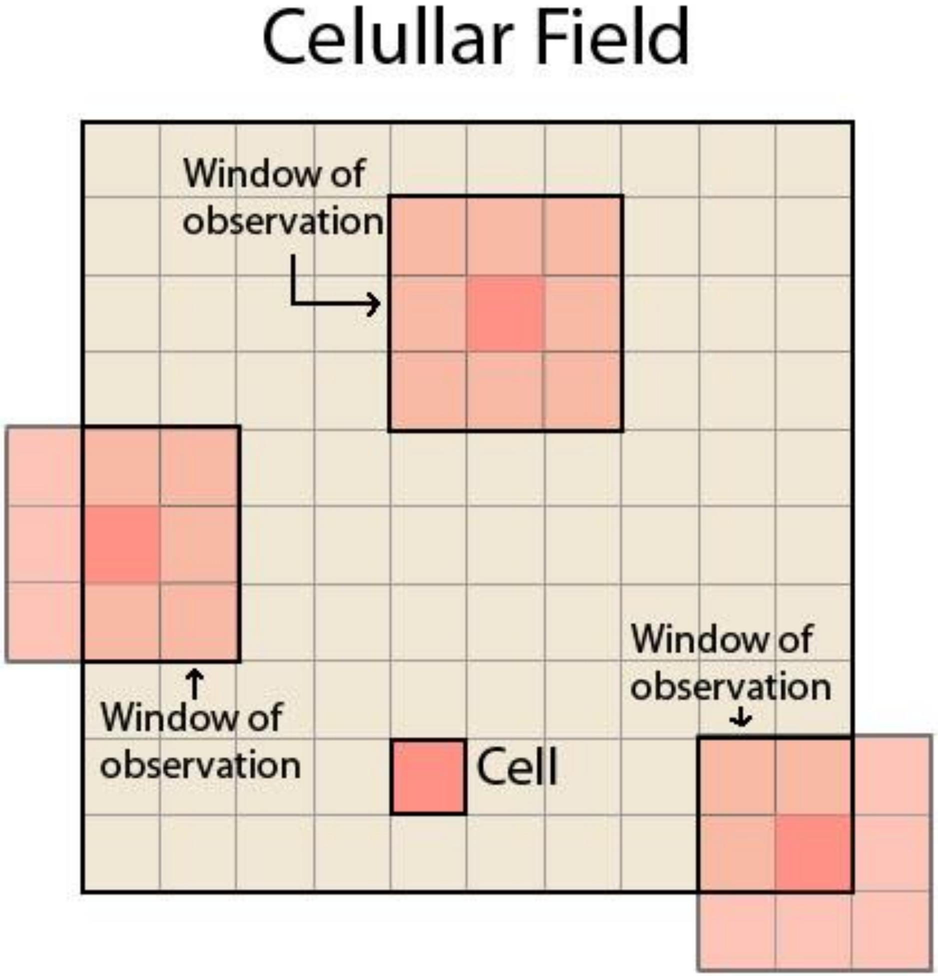
$w: S \rightarrow [0,1],\ w_{ij} = w(i, j)$, such that

$$\sum_{(i,j) \in S} w_{ij} = 1$$

(the influence degree of (i,j)-cell of S to the central one)

The $(\mathbf{i}, \mathbf{j})$-cell window of observation

$$S_{ij} = \{(i, j) + S\} \cap T$$



To each $(i, j) \in T$ we assign

vitality probability

$p_{ij} \in [0,1]$, and

(i, j)-cell state:

Bernoulli random variable

$$\xi_{ij} = \begin{cases} 1, & with\ probability\ p_{ij}, \\ 0, & with\ probability\ 1 - p_{ij}. \end{cases}$$

1 is alive

0 is dead

Evolution

Evolution of the cells states is described by means of vitality probabilities p_{ij}.

<u>Model 1</u>

Put ${p_{ij}}^{(0)} = p_{ij}$, and

$$ {p_{ij}}^{(n)} = \sum_{(k,l)\in S_{ij}} w_{kl}\, {p_{kl}}^{(n-1)}, $$

$\mathrm{n} \in \boldsymbol{N}$.

The main aim is to investigate the limiting behavior of p_{ij}.

Theorem 1 (E. Tuzhilina)

For any $(i, j) \in T$ we have

$$\lim_{n \to \infty} p_{ij}{}^{(n)} = 0 \,.$$

Remark.

Person is mortal with probability 1 (in Model 1).

Visualization of the result

m=10

r =1

Weight coefficients matrix (w_{ij})= $\begin{pmatrix} 0 & 0 & 0.35 \\ 0 & 0.5 & 0.15 \\ 0 & 0 & 0 \end{pmatrix}$

Vitality probability matrix (p_{ij})

0.03	0.66	0.98	0.87	0.57	0.88	0.51	0.79	0.4	0.91
0.48	0.08	0.02	0.39	0.25	0.91	0.49	0.69	0.4	0.39
0.57	0.91	0.39	0.29	0.25	0.52	0.89	0.27	0.74	0.35
0.32	0.91	0.62	0.81	0.17	0.94	0.95	0.91	0.14	0.85
0.56	0.81	0.3	0.99	0.3	0.92	0.41	0.12	0.03	0.65
0.74	0.37	0.86	0.03	0.08	0.73	0.29	0.34	0.94	0.66
0.12	0.74	0.75	0.5	0.26	0.67	0.69	0.54	0.08	0.25
0.52	0.56	0.44	0.04	0.65	0.88	0.7	0.9	0.65	0.05
0.16	0.32	0.62	0.66	0.22	0.45	0.09	0.8	0.26	0.62
0.2	0.34	0.39	0.7	0.47	0.92	0.16	0.02	0.6	0.26

To visualize, represent (p_{ij}) as a coloured table dark blue is 1

white is 0.

0.03	0.66	0.98	0.87	0.57	0.88	0.51	0.79	0.4	0.91
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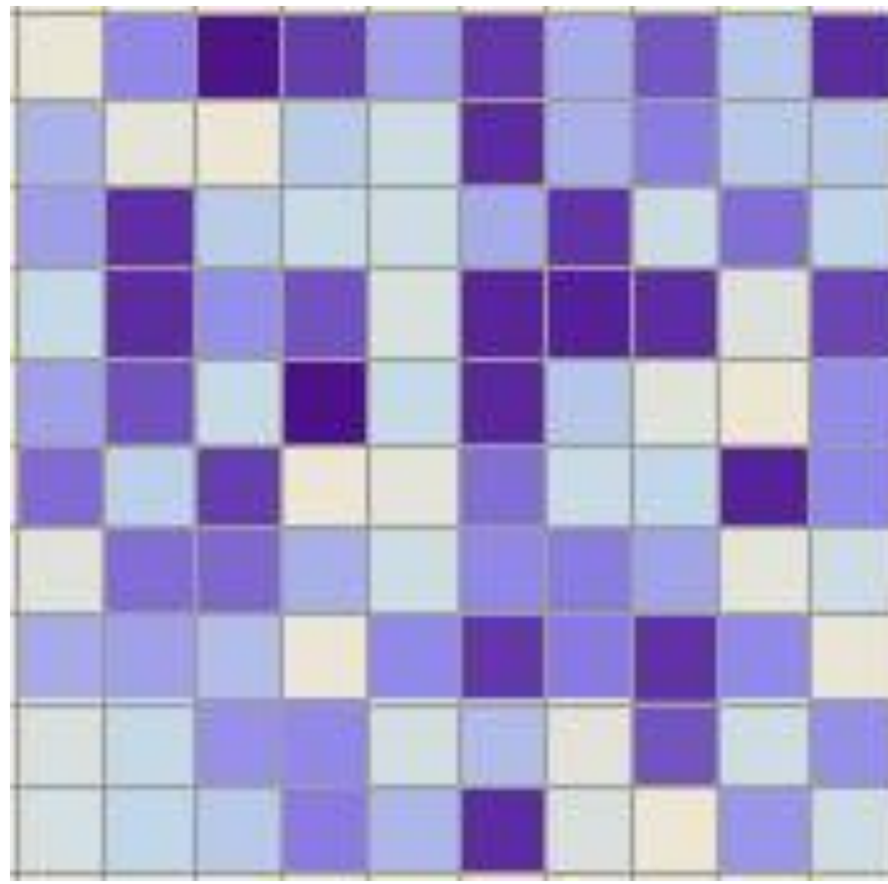
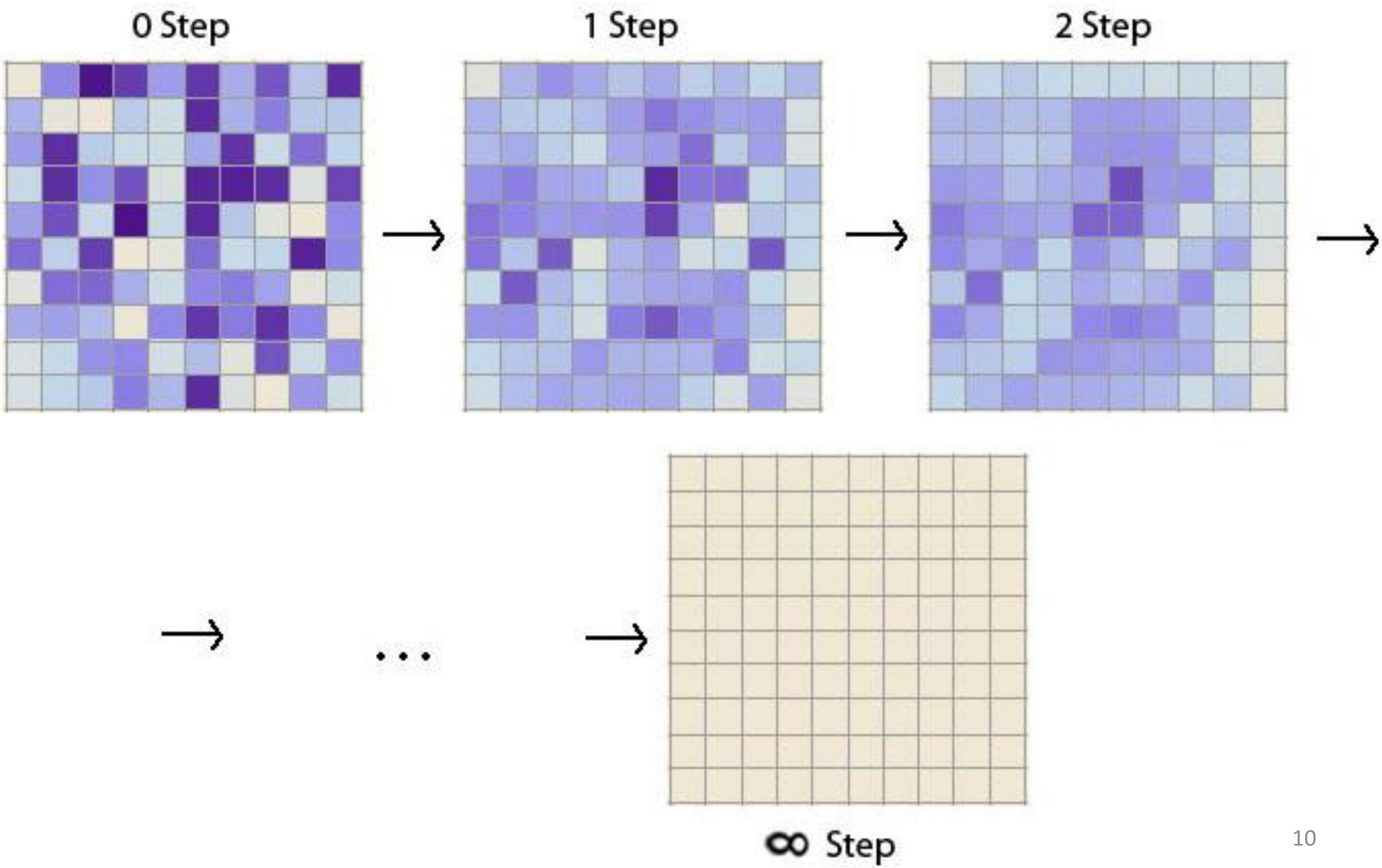


Illustration



Remark.

This phenomenon appears due to “boundary effect”, when some of the observation windows for cells close to boundary are trimmed.

Sketch of the proof

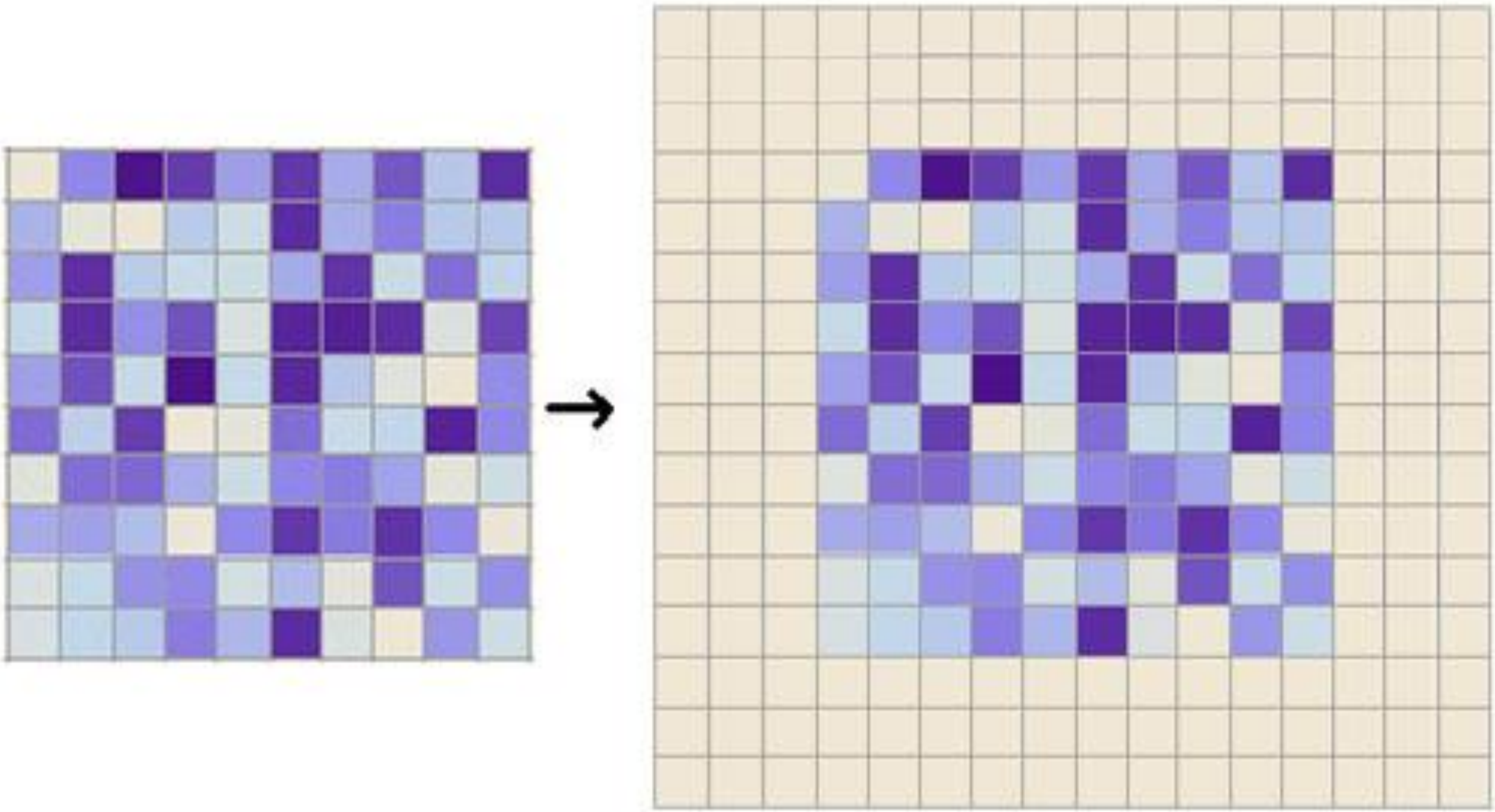
Extend T to $T' = \mathbb{Z}^2$

for each $(i, j) \in T'$ we put

$$\widetilde{p_{ij}} = \begin{cases} p_{ij}, & if\ (i,j) \in \mathrm{T}, \\ 0, & if (i,j) \in \mathrm{T}' \backslash \mathrm{T}. \end{cases}$$

$$\mathrm{S}_{ij} = (i, j) + S$$

Illustration



Evolution in this case characterized by values $\widetilde{p_{ij}}$

$$\widetilde{p_{ij}}^{(0)} = \widetilde{p_{ij}}$$

$$\widetilde{p_{ij}}^{(n)} = \sum_{(k,l)\in \mathrm{S}_{ij}} w_{kl}\, \widetilde{p_{kl}}^{(n-1)}$$

Notice that

$$\sum_{(k,l)\in T'} \widetilde{p_{kl}}^{(n)} = \sum_{(k,l)\in T'} \widetilde{p_{kl}}^{(n-1)}$$

Observe that $\widetilde{p_{ij}}^{(n)}$ is a linear function defined by $\widetilde{p_{kl}}^{(n-1)}$ where $(k, l) \in S_{ij}$.

Thus there is a linear operator

$$\tilde{P}^{(n)} = A\tilde{P}^{(n-1)}$$

where $\tilde{P}^{(n)}$=($\widetilde{p_{ij}}^{(n)}$),
the operator A depends only on weight coefficients w_{ij}.

If some of $\widetilde{p_{ij}}$ is not zero then there exist a step s and a cell $(k, l) \in T' \backslash \mathrm{T}$ for which $\widetilde{p_{kl}}^{(s)} \neq 0$.

For $x = (x_1, \dots , x_n)$ we put

$$||x|| = \sum_{i=1}^{n} x_i$$

Thus $||A^s|| < 1$ and the mapping will be contractive. We come to desired result.

Equivalent formulation

Replace T' by

$\{-m-r, \ldots, m+r\} \times \{-m-r, \ldots, m+r\}$

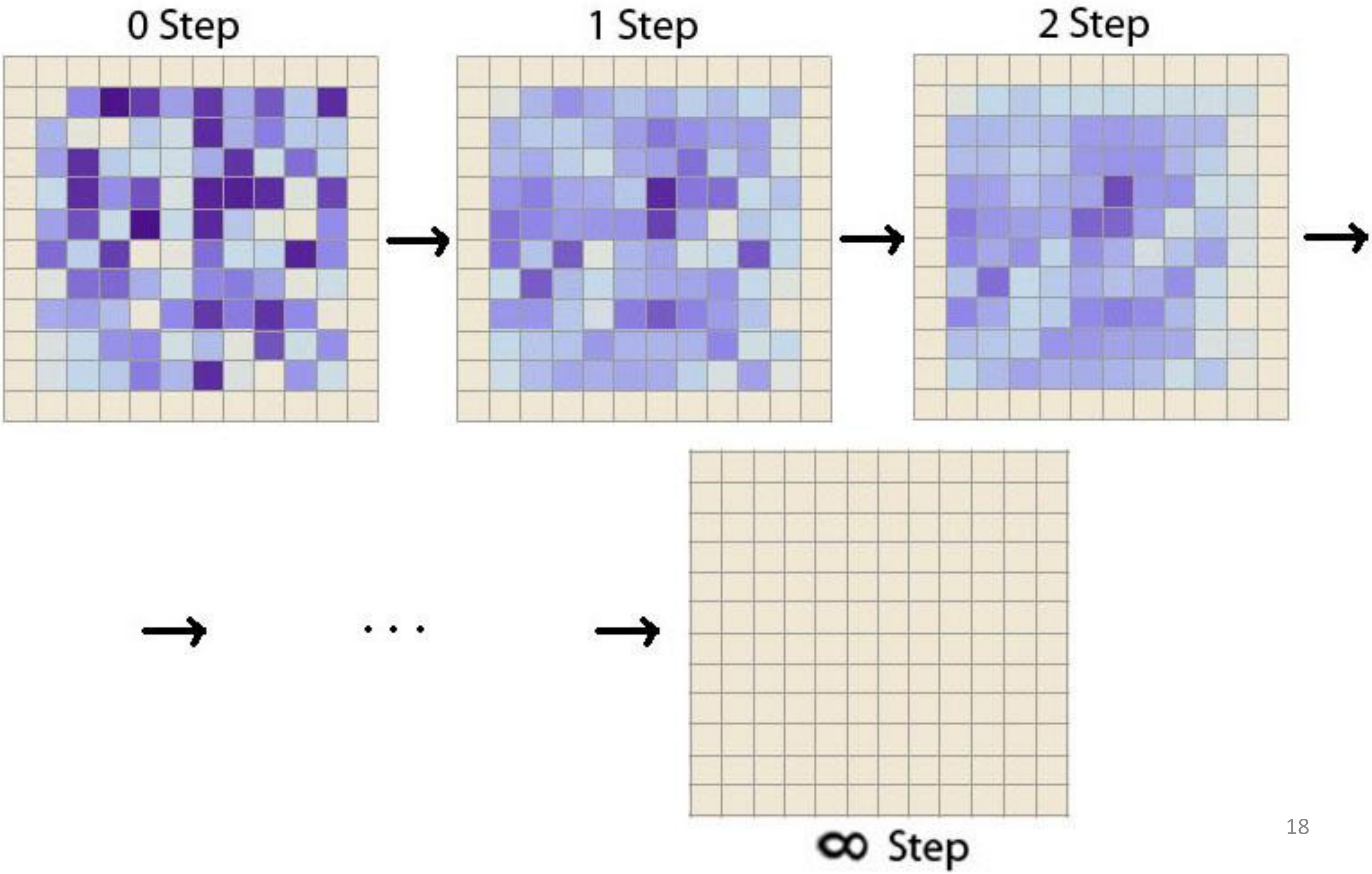
Evolution process works only with cells $(i, j) \in \mathbf{T}$.

Remark.

One has $\widetilde{p_{ij}}^{(n)} = p_{ij}^{(n)}$ for $(i, j) \in \mathbf{T}$, thus

$\lim_{n\to\infty} \widetilde{p_{ij}}^{(n)} = 0$ for every cells.

Illustration



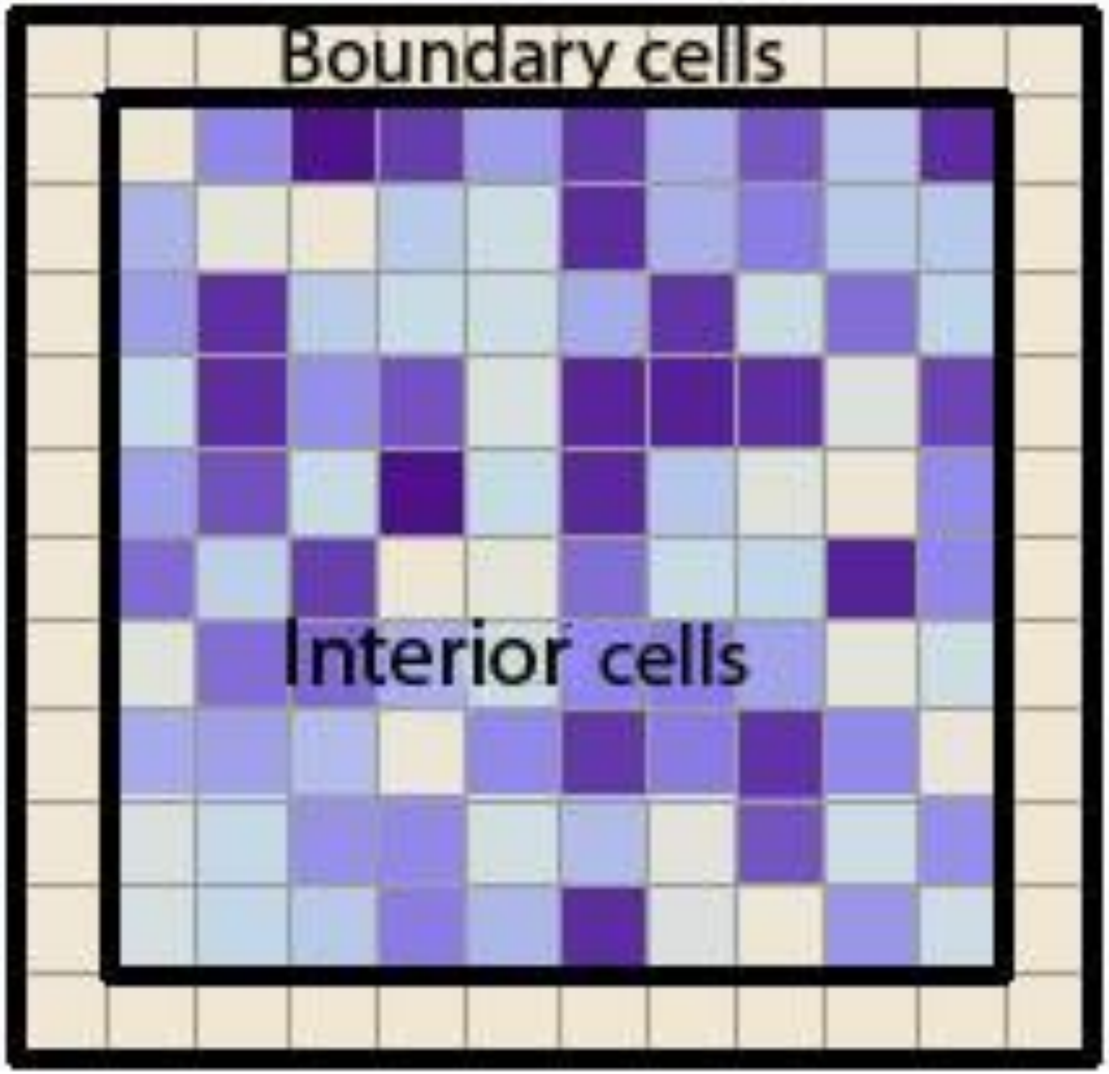
Definition

Interior cells:

$(i, j) \in \mathrm{T} \subseteq T'$

Boundary cells:

$(i, j) \in \mathrm{T}' \backslash T$



Model 2

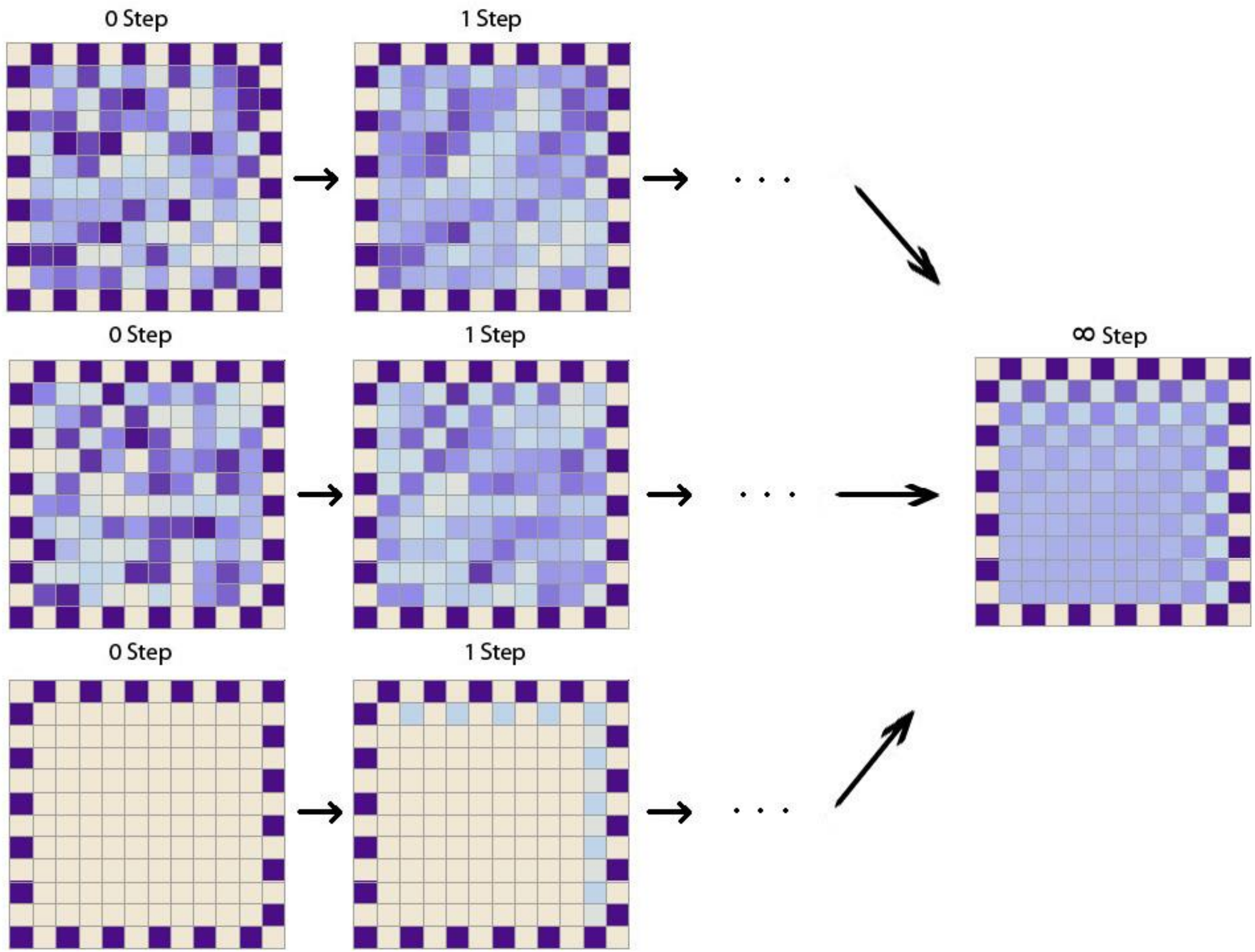
Redefine $\widetilde{p_{ij}}$:

for each $(i, j) \in \mathrm{T}'\backslash\mathrm{T}$ we change $\widetilde{p_{ij}} = 0$ to arbitrary values from [0,1].

Remark.

Now the limit of $\widetilde{p_{ij}}^{(n)}$ can be nonzero.

Example: $\widetilde{p_{ij}} = i + j (mod\ 2)$ for $(i, j) \in T' \backslash T$.



Theorem 2 (E. Tuzhilina)

The limits of $\widetilde{p_{ij}}^{(n)}$ depend only on boundary values $\widetilde{p_{kl}}$ where $(k, l) \in \mathrm{T}' \backslash \mathrm{T}$ and do not depend on the interior ones.

Remark.

Your chances to survive are completely determined by your surroundings.

How to calculate the limiting $\widetilde{p_{ij}}^{(n)}$

Notice that $\widetilde{p_{ij}}^{(n)}$ is a linear function on $\widetilde{p_{kl}}^{(n-1)}$ and the boundary $\widetilde{p_{qs}}$. Thus there is an affine operator

$$P^{(n)} = AP^{(n-1)} + B,$$

where $P^{(n)}$=($\widetilde{p_{ij}}^{(n)}$), A depends only on weight coefficients w_{ij} where $(i, j) \in S$, and B depends on w_{ij} and boundary values $\widetilde{p_{qs}}$ where $(i, j) \in \mathrm{T}' \backslash \mathrm{T}$.

Theorem 3 (E. Tuzhilina)

Suppose that some of $\widetilde{p_{ij}}$ is not zero and $w_{ij} \in [0,1]$ for all $(i, j) \in S$. Then there exists a step s such that $||A^s||<1$, and

$$\lim_{n\to\infty} P^{(n)} = (E - A^s)^{-1} P^{(0)} .$$

(See illustration in Wolfram Mathematica)

General model

Let X, Z be disjoined sets and $Y = X \sqcup Z$

$w : X \times Y \to [0,1],\ w_x(y) = w(x, y),$

$\forall x \in X\ \sum_{y \in Y} w_x(y) = 1$

$p: Y \to [0,1],\ \ p_y = p(y)$

$\forall y \in Y$

$$\xi_y = \begin{cases} 1,\ with\, probability\, p_y, \\ 0,\ \ with\, probability\, 1 - p_y. \end{cases}$$

$$p_y^{(0)} = p_y$$

$$p_y^{(n)} = \begin{cases} \sum_{y \in Y} p_y^{(n-1)} w_x(y), & for\ y \in X, \\ p_y, & for\ y \in Z. \end{cases}$$

Remark.

Theorem 3 holds for the general model.

In particular, the same technique can be applied to

- arbitrary dimension
- arbitrary placement of boundary and interior cells

References

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Thank you for attention