Exercise 1  \((1+2+2+3+3)\)

Let \(H \in (0, 1)\) be some constant. A stochastic process \(\{B^{(H)}_t, t \geq 0\}\) is said to be a fractional Brownian motion (fBm) of Hurst index \(H\) if it is a Gaussian process with \(\mathbb{E}B^{(H)}_t = 0\) for each \(t \geq 0\) and if its covariance function fulfills \(\text{Cov}(B^{(H)}_t, B^{(H)}_s) = 1/2(t^{2H} + s^{2H} - |t-s|^{2H})\) for all \(s, t \geq 0\).

a) Show that \(B^{(H)}_0 = 0\) a.s. for each \(H \in (0, 1)\).

b) Let \(s, t \geq 0\). Show that \(B^{(H)}_{t+s} - B^{(H)}_s \overset{D}{=} B^{(H)}_t\) for each \(H \in (0, 1)\).

c) Show that the fBm is self-similar for each Hurst-index \(H \in (0, 1)\), i.e., show that for each \(a > 0\) there exists a constant \(b \in \mathbb{R}\) such that \(\{B^{(H)}_{at}, t \geq 0\} \overset{D}{=} \{bB^{(H)}_t, t \geq 0\}\).

d) A stationary stochastic process \(\{Y_n, n \in \mathbb{N}\}\) in discrete time is said to be long-range dependent if \(\lim_{n \to \infty} c^{-n} n^\alpha \text{Cov}(Y_0, Y_n) = 1\) for some \(c \in \mathbb{R}\) and \(\alpha \in (0, 1)\). Define \(\{Y_n, n \in \mathbb{N}\}\) by \(Y_n = B^{(H)}_{n+1} - B^{(H)}_n\) for each \(n \in \mathbb{N}\). Show that \(\{Y_n, n \in \mathbb{N}\}\) is long-range dependent for each \(H \in (1/2, 1)\).

e) Write an R-code (or Matlab-code) in order to simulate an approximation of \(\{B^{(H)}_t, t \in [0, 1]\}\) for \(H \in \{1/10, 1/2, 9/10\}\) by simulating the process \(\{Y_t, t \in [0, 1]\}\) defined by
\[
Y_{k/500} = B^{(H)}_{k/500}
\]
for each \(k \in \{0, 1, \ldots, 500\}\) and
\[
Y_t = B^{(H)}_{k/500} + (500t - k)(B^{(H)}_{(k+1)/500} - B^{(H)}_{k/500})
\]
for each \(t \in (k/500, (k+1)/500)\), where \(k \in \{0, 1, \ldots, 499\}\). Hand in your code and one realization for each for \(H \in \{1/10, 1/2, 9/10\}\).

Hint: Use the command \texttt{mvrnorm} in package \texttt{MASS} for programming in R and use the command \texttt{mvnrnd} for programming in Matlab.

Remark: In Exercise Sheet 8 we defined the non-degenerate multivariate normal distribution by its density function. Here we need the general case of a multivariate normal distribution, where the covariance matrix is not necessarily positive-definite, but positive semi-definite,
i.e.: Let $\mu = (\mu_1, \ldots, \mu_n)^\top \in \mathbb{R}^n$ and $K = (k_{i,j})_{i,j=1,\ldots,n}$ be a symmetric and positive semi-definite $n \times n$-matrix. The random vector $Z = (Z_1, \ldots, Z_n)^\top$ is said to be multivariate normal distributed with mean vector $\mu$ and covariance matrix $K$ if its characteristic function is given by $\varphi(t) = \exp \left( it^\top \mu - \frac{1}{2} t^\top K t \right)$ for all $t \in \mathbb{R}^n$. Recall, that a stochastic process $\{V_t, t \geq 0\}$ is said to be a Gaussian process if the random vector $(V_{t_1}, \ldots, V_{t_n})$ is multivariate normal for each $n \in \mathbb{N}$ and for all $0 \leq t_1, \ldots, t_n < \infty$.

**Exercise 2** (2)

Let $c > 0$ be arbitrary and let $\nu$ be an arbitrary Lévy measure. Let $N$ be a random variable with $N \sim \text{Poi}(\nu([-c,c]^c))$ and let $U_1, U_2, \ldots$ be a sequence of i.i.d. random variables such that $\Pr(U_1 \in B) = \nu(B \cap [-c,c]^c)/\nu([-c,c]^c)$ for each Borel set $B \subset \mathbb{R}$. Define the random variable $Y = \sum_{k=1}^N U_k$. Show that the characteristic function of $Y$ is given by

$$\varphi(s) = \exp \left( \int_{[-c,c]^c} (\exp(isy) - 1)\nu(dy) \right).$$

Note that $Y$ is said to have a compound Poisson distribution.

**Exercise 3** (2)

Let $X_1, \ldots, X_n$ be independent and infinitely divisible random variables and let $a_1, \ldots, a_n \in \mathbb{R}$ be arbitrary numbers. Show that $\sum_{k=1}^n a_k X_k$ is an infinitely divisible random variable.

**Exercise 4** (3+3)

Show that the following random variables $X, Y$ are infinitely divisible.

a) Let $r \in \mathbb{N}$ and $p \in (0, 1)$. Define the random variable $X : \Omega \to \mathbb{N}_0$ by

$$\Pr(X = k) = \binom{k + r - 1}{k} p^k (1 - p)^r,$$

for each $k \in \mathbb{N}_0$.

*Hint: Consider the sum of i.i.d. geometric distributed random variables.*

b) Let $N \in \mathbb{N}$. Define the random variable $Y$ by its characteristic function

$$\varphi_Y(s) = \exp \left( i \sum_{k=1}^N \sin(sk) + \sum_{k=1}^N (\cos(sk) - 1) \right).$$