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Winter term 2014/15

Stochastics II
Exercise Sheet 10

Deadline: January 7, 2015 at 4pm before the exercises

Exercise 1 (1+2+2+3+3)

Let $H \in (0,1)$ be some constant. A stochastic process $\{B_t^{(H)}, t \geq 0\}$ is said to be a fractional Brownian motion (fBm) of Hurst index H if it is a Gaussian process with $\mathbb{E}B_t^{(H)} = 0$ for each $t \geq 0$ and if its covariance function fulfills $\operatorname{Cov}(B_t^{(H)}, B_s^{(H)}) = 1/2(t^{2H} + s^{2H} - |t-s|^{2H})$ for all $s,t \geq 0$.

a) Show that $B_0^{(H)} = 0$ a.s. for each $H \in (0,1)$.

b) Let $s,t \geq 0$. Show that $B_{t+s}^{(H)} - B_s^{(H)} \stackrel{D}{=} B_t^{(H)}$ for each $H \in (0,1)$.

c) Show that the fBm is self-similar for each Hurst-index $H \in (0,1)$, i.e., show that for each $a > 0$ there exists a constant $b \in \mathbb{R}$ such that

$$\{B_{at}^{(H)}, t \geq 0\} \stackrel{D}{=} \{bB_t^{(H)}, t \geq 0\}.$$

d) A stationary stochastic process $\{Y_n, n \in \mathbb{N}\}$ in discrete time is said to be long-range dependent if $\lim_{n\to\infty} c^{-1} n^{\alpha} \operatorname{Cov}(Y_0, Y_n) = 1$ for some $c \in \mathbb{R}$ and $\alpha \in (0,1)$. Define $\{Y_n, n \in \mathbb{N}\}$ by $Y_n = B_{n+1}^{(H)} - B_n^{(H)}$ for each $n \in \mathbb{N}$. Show that $\{Y_n, n \in \mathbb{N}\}$ is long-range dependent for each $H \in (1/2, 1)$.

e) Write an R-code (or Matlab-code) in order to simulate an approximation of $\{B_t^{(H)}, t \in [0,1]\}$ for $H \in \{1/10, 1/2, 9/10\}$ by simulating the process $\{Y_t, t \in [0,1]\}$ defined by

$$Y_{k/500} = B_{k/500}^{(H)}$$

for each $k \in \{0, 1, \ldots, 500\}$ and

$$Y_t = B_{k/500}^{(H)} + (500t - k)(B_{(k+1)/500}^{(H)} - B_{k/500}^{(H)})$$

for each $t \in (k/500, (k+1)/500)$, where $k \in \{0, 1, \ldots, 499\}$. Hand in your code and one realization for each for $H \in \{1/10, 1/2, 9/10\}$.
Hint: Use the command mvrnorm *in package MASS for programming in R and use the command* mvnrnd *for programming in Matlab.*

Remark: In Exercise Sheet 8 we defined the non-degenerate multivariate normal distribution by its density function. Here we need the general case of a multivariate normal distribution, where the covariance matrix is not necessarily positive-definite, but positive semi-definite, *i.e.: Let $\mu = (\mu_1, \dots, \mu_n)^\top \in \mathbb{R}^n$ and $K = (k_{i,j})_{i,j=1,\dots,n}$ be a symmetric and positive semi-definite $n \times n$-matrix. The random vector $Z = (Z_1, \dots, Z_n)^\top$ is said to be multivariate normal distributed with mean vector μ and covariance matrix K if its characteristic function is given by $\varphi(t) = \exp\left(it^\top \mu - \frac{1}{2} t^\top K t\right)$ for all $t \in \mathbb{R}^n$. Recall, that a stochastic process $\{V_t, t \geq 0\}$ is said to be a Gaussian process if the random vector $(V_{t_1}, \dots, V_{t_n})$ is multivariate normal for each $n \in \mathbb{N}$ and for all $0 \leq t_1, \dots, t_n < \infty$.*

Exercise 2 (2)

Let $c > 0$ be arbitrary and let ν be an arbitrary Lévy measure. Let N be a random variable with $N \sim \mathrm{Poi}(\nu([-c,c]^c))$ and let $U_1, U_2, \dots$ be a sequence of i.i.d. random variables such that $\mathbb{P}(U_1 \in B) = \nu(B \cap [-c,c]^c)/\nu([-c,c]^c)$ for each Borel set $B \subset \mathbb{R}$. Define the random variable $Y = \sum_{k=1}^N U_k$. Show that the characteristic function of Y is given by

$$\varphi(s) = \exp\left(\int_{[-c,c]^c} (\exp(isy) - 1)\nu(\mathrm{d}y)\right).$$

Note that Y is said to have a compound Poisson distribution.

Exercise 3 (2)

Let $X_1, \dots, X_n$ be independent and infinitely divisible random variables and let $a_1, \dots, a_n \in \mathbb{R}$ be arbitrary numbers. Show that $\sum_{k=1}^n a_k X_k$ is an infinitely divisible random variable.

Exercise 4 (3+3)

Show that the following random variables X, Y are infinitely divisible.

a) Let $r \in \mathbb{N}$ and $p \in (0,1)$. Define the random variable $X : \Omega \to \mathbb{N}_0$ by

$$\mathbb{P}(X = k) = \binom{k+r-1}{k} p^k (1-p)^r,$$

for each $k \in \mathbb{N}_0$.
Hint: Consider the sum of i.i.d. geometric distributed random variables.

b) Let $N \in \mathbb{N}$. Define the random variable Y by its characteristic function

$$\varphi_Y(s) = \exp\left(i \sum_{k=1}^N \sin(sk) + \sum_{k=1}^N (\cos(sk) - 1)\right).$$