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Winter term 2014/15

Stochastics II
Exercise Sheet 11

Deadline: January 14, 2015 at 4pm before the exercises

Exercise 1 (5+3)

Let $X : \Omega \to \mathbb{R}$ be a real-valued random variable with characteristic function φ_X. Show the following statements:

a) If X is infinitely divisible, then it holds $\varphi_X(t) \neq 0$ for all $t \in \mathbb{R}$.
Hint: Show that $\varphi_X(s) = (\varphi_n(s))^n$ *for each* $s \in \mathbb{R}$ *implies* $\lim_{n\to\infty} |\varphi_n(s)| = \mathbb{I}(\varphi(s) \neq 0)$ *for each* $s \in \mathbb{R}$*. Moreover, you may assume the following without proof: Let* $0 < \theta < 2\pi$ *and let* $\{c_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{C}$ *with* $\lim_{n\to\infty} c_n = \exp(i\theta)$*. Then the limit* $\lim_{n\to\infty} c_n^n$ *does not exist.*

b) There exists a real-valued random variable $Y : \Omega \to \mathbb{R}$ such that its characteristic function φ_Y fulfills $\varphi_Y(t) \neq 0$ for all $t \in \mathbb{R}$, but Y is not infinitely divisible.
Hint: Consider a random variable Y *such that* $\mathbb{P}(Y \in \{-1, 0, 1\}) = 1$*.*

Exercise 2 (4)

Let $\{X_t, t \geq 0\}$ be a Lévy process with characteristic triplet (a, b, ν), where ν is a finite measure on $\mathbb{R}$, i.e. $\nu(\mathbb{R}) < \infty$. Show that there exist a Wiener process $\{W_t, t \geq 0\}$, a compound Poisson process $\{Z_t, t \geq 0\}$ independent of $\{W_t, t \geq 0\}$ and constants $\mu, c \in \mathbb{R}$ with $c \neq 0$ such that the following holds:

$$\{X_t, t \geq 0\} \stackrel{D}{=} \{cW_t + \mu t + Z_t, t \geq 0\}.$$

Exercise 3 (1+3+2)

Let $\{X_t, t \geq 0\}$ be a Lévy process and $b, p > 0$ such that $X_t \sim \Gamma(b, pt)$ for each $t > 0$, i.e. the probability density function of X_t is given by $f_{X_t}(x) = b^{pt} x^{pt-1} \exp(-bx)/\Gamma(pt)$, for each $t \geq 0$. Note that $\{X_t, t \geq 0\}$ is said to be a gamma process with parameters b and p.

a) Show that $\int_0^\infty \min\{1, y\} p y^{-1} \exp(-by) \mathrm{d}y < \infty$

b) Show that $\{X_t, t \geq 0\}$ is a subordinator with

$$\mathbb{E} \exp(-uX_t) = \exp\left(-t \int_0^\infty \frac{p(1 - \exp(-uy))}{y \exp(by)} \mathrm{d}y\right)$$

for all $u \geq 0$.

c) Let $b = 10, p = 100$. Write an R-code (or Matlab-code) in order to simulate an approximation of $\{X_t, t \geq 0\}$ on the interval $[0, 1]$ by simulating the process $\{Y_t, t \in [0, 1]\}$ defined by

$$Y_{k/100} = X_{k/100}$$

for each $k \in \{0, 1, \dots, 100\}$ and

$$Y_t = X_{k/100} + (100t - k)(X_{(k+1)/100} - X_{k/100})$$

for each $t \in (k/100, (k+1)/100)$, where $k \in \{0, 1, \dots, 99\}$. Hand in your code and a plot of one realization.