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Winter term 2014/15

Stochastics II
Exercise Sheet 12

Deadline: January 21, 2015 at 4pm before the exercises

Exercise 1 (2+2+4)

Let $X : \Omega \to \mathbb{R}$ be an α-stable distribution with $\alpha \in (0, 2]$.

a) Show that X is infinitely divisible.

b) Let from now on $X \stackrel{D}{=} -X$. Show that $(\varphi_X(s))^n = \varphi_X(n^{1/\alpha}s)$ for each $s \in \mathbb{R}$ and each $n \in \mathbb{N}$, where φ_X denotes the characteristic function of X.

c) Use part b) in order to show that there exists a constant c_α such that $\varphi_X(s) = \exp(-c_\alpha|s|^\alpha)$.

Exercise 2 (2+2+3+3+2)

Let $X, Y : \Omega \to \mathbb{R}$ be random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}|X| < \infty, \mathbb{E}|Y| < \infty$ and $\mathbb{E}|XY| < \infty$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Show the following properties of conditional expectation.

a) $\mathbb{E}(aX + Y \mid \mathcal{G}) = a\mathbb{E}(X \mid \mathcal{G}) + \mathbb{E}(Y \mid \mathcal{G})$ for each $a \in \mathbb{R}$.

b) $\mathbb{E}(X \mid \mathcal{G}) \leq \mathbb{E}(Y \mid \mathcal{G})$ if $X \leq Y$ a.s.

c) Let $0 \leq X_1 \leq X_2 \leq \ldots$ be a sequence of non-negative random variables on $(\Omega, \mathcal{F}, P)$ with $\lim_{n\to\infty} X_n = X$ a.s. Then it holds $\lim_{n\to\infty} \mathbb{E}(X_n \mid \mathcal{G}) = \mathbb{E}(X \mid \mathcal{G})$.

d) $\mathbb{E}(XY \mid \mathcal{G}) = Y\mathbb{E}(X \mid \mathcal{G})$ if Y is measurable w.r.t. $\mathcal{G}$.

e) $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}_2) \mid \mathcal{G}_1) = \mathbb{E}(X \mid \mathcal{G}_1)$ if $\mathcal{G}_1$ and G_2 are sub-σ-algebras of $\mathcal{F}$ with $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$.

Exercise 3 (2+2)

Let $X, Y : \Omega \to \mathbb{R}$ be random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}X^2 < \infty$. Define the conditional variance by $\operatorname{Var}(X \mid Y) = \mathbb{E}((X - \mathbb{E}(X \mid Y))^2 \mid Y)$.

a) Show that $\operatorname{Var}(X) = \mathbb{E}\operatorname{Var}(X \mid Y) + \operatorname{Var}\mathbb{E}(X \mid Y)$.

b) Let $N \sim \operatorname{Bin}(n, p)$ for $n \in \mathbb{N}, p \in (0, 1)$ and let $U_1, \ldots, U_n$ be a sequence of i.i.d. Poisson distributed random variables with $U_1 \sim \operatorname{Poi}(\lambda)$ for some $\lambda > 0$. Compute $\operatorname{Var}\left(\sum_{i=1}^N U_i\right)$ using part a).

Exercise 4

Let $T, T' : \Omega \to [0, \infty]$ be arbitrary stopping times and let $\alpha \in (1, \infty)$ be arbitrary. Show that the following random variables are stopping times as well: $\min\{T, T'\}, \max\{T, T'\}, T+T', \alpha T$.