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Winter term 2014/15

Stochastics II
Exercise Sheet 13

Deadline: January 28, 2015 at 4pm before the exercises

Exercise 1 (3+3)

Let $\{\mathcal{F}_t, t \geq 0\}$ be a filtration on a probability space $(\Omega, \mathcal{F}, P)$. Let τ be a stopping time. We define the σ-algebra $\mathcal{F}_\tau$ by $\mathcal{F}_\tau = \{B \in \mathcal{F} : B \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for each } t \geq 0\}$. Moreover, let S and T be stopping times w.r.t. $\{\mathcal{F}_t, t \geq 0\}$. Show the following results:

a) $A \cap \{S \leq T\} \in \mathcal{F}_T$ for each $A \in \mathcal{F}_S$

b) $\mathcal{F}_{\min\{S,T\}} = \mathcal{F}_S \cap \mathcal{F}_T$

Exercise 2 (2)

Let $\{X_t, t \geq 0\}$ be a martingale. Show that $\mathbb{E}X_t = \mathbb{E}X_0$ for each $t \geq 0$.

Exercise 3 (4+4)

a) Let $\{X_t, t \geq 0\}$ be an inhomogeneous Poisson process with intensity function $\lambda_t \geq 0$ for each $t \geq 0$. Define the process $\{Y_t, t \geq 0\}$ by $Y_t = X_t - \int_0^t \lambda(s)\mathrm{d}s$. Show that $\{Y_t, t \geq 0\}$ is a martingale with respect to the intrinsic filtration $\{\mathcal{F}_t^X\}$.
Remark: Recall that an inhomogeneous Poisson process is a Cox process where the intensity function λ_t is deterministic.

b) Let $\{X_t, t \geq 0\}$ be a Lévy process with $\mathbb{E}|X_1| < \infty$. Define the process $\{Y_t, t \geq 0\}$ by $Y_t = X_t - t\mathbb{E}X_1$. Show that $\{Y_t, t \geq 0\}$ is a martingale with respect to the intrinsic filtration $\{\mathcal{F}_t^X\}$.

Exercise 4 (4 extra points)

Let X be a random variable with $\mathbb{E}|X| < \infty$ and let $\{X_n, n \geq 1\}$ be a sequence of random variables with $\lim_{n\to\infty} \mathbb{E}|X_n - X| = 0$. Show that $\{X_n, n \geq 1\}$ is uniformly integrable.