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Winter term 2014/15

Stochastics II
Exercise Sheet 14

Deadline: February 4, 2015 at 4pm before the exercises

Exercise 1 (5+5 extra points)

Let $\{X_t, t \geq 0\}$ be a Wiener process and let $\{N_t, t \geq 0\}$ be a homogeneous Poisson process with intensity $\lambda > 0$.

a) Define the process $\{Y_t, t \geq 0\}$ by $Y_t = \exp(X_t)$. Show that $\{Y_t, t \geq 0\}$ is a sub-martingale with respect to the intrinsic filtration $\{\mathcal{F}_t^X\}$ of $\{X_t, t \geq 0\}$.

b) Define the process $\{Y_t, t \geq 0\}$ by $Y_t = N_t - \log(N_t + 1)$. Show that $\{Y_t, t \geq 0\}$ is a sub-martingale with respect to the intrinsic filtration $\{\mathcal{F}_t^N\}$ of $\{N_t, t \geq 0\}$.

Exercise 2 (10 extra points)

Let $\{X_t, t \geq 0\}$ be adapted, cádlág and a martingale. Show that

$$\mathbb{P}\left(\sup_{0 \leq v \leq t} |X_v| > x\right) \leq \frac{\mathbb{E}|X_t|}{x},$$

for all $x > 0$ and $t \geq 0$.

Exercise 3 (5+10 extra points)

Let $\{\mathcal{F}_n, n \in \mathbb{N}\}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X_1, X_2, \ldots : \Omega \to \mathbb{R}$ be a sequence of random variables such that X_n is measurable with respect to $\mathcal{F}_n$ for each $n \in \mathbb{N}$. Then, $\{X_n, n \in \mathbb{N}\}$ is said to be a discrete-time martingale if $\mathbb{E}(X_{n+1} \mid X_n) = X_n$. A discrete stopping time with respect to $\{\mathcal{F}_n, n \in \mathbb{N}\}$ is a random variable $T : \Omega \to \mathbb{N} \cup \{\infty\}$ such that $\{T \leq n\} \in \mathcal{F}_n$ for each $n \in \mathbb{N} \cup \{\infty\}$ where $\mathcal{F}_\infty = \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n)$.

a) Let $g : [0, \infty) \to [0, \infty)$ be a monotonously increasing function with $\lim_{x \to \infty} g(x)/x = \infty$. Show that $\{X_n, n \in \mathbb{N}\}$ is uniformly integrable if $\sup_{n \in \mathbb{N}} \mathbb{E}g(|X_n|) < \infty$.

b) Let $\{X_n, n \in \mathbb{N}\}$ be a discrete-time martingale and let T be a discrete stopping time such that $\mathbb{E}|X_T| < \infty$ and $\lim_{n \to \infty} \mathbb{E}(|X_n| \mathbb{I}(T > n)) = 0$. Show that the sequence of random variables $X_{\min\{1,T\}}, X_{\min\{2,T\}}, \ldots$ is uniformly integrable.