Exercise 1  (5+5 extra points)

Let \{X_t, t \geq 0\} be a Wiener process and let \{N_t, t \geq 0\} be a homogeneous Poisson process with intensity \(\lambda > 0\).

a) Define the process \(\{Y_t, t \geq 0\}\) by \(Y_t = \exp(X_t)\). Show that \(\{Y_t, t \geq 0\}\) is a sub-martingale with respect to the intrinsic filtration \(\mathcal{F}^X_t\) of \(\{X_t, t \geq 0\}\).

b) Define the process \(\{Y_t, t \geq 0\}\) by \(Y_t = N_t - \log(N_t + 1)\). Show that \(\{Y_t, t \geq 0\}\) is a sub-martingale with respect to the intrinsic filtration \(\mathcal{F}^N_t\) of \(\{N_t, t \geq 0\}\).

Exercise 2  (10 extra points)

Let \(\{X_t, t \geq 0\}\) be adapted, cádlág and a martingale. Show that

\[
P\left( \sup_{0 \leq v \leq t} |X_v| > x \right) \leq \frac{E|X_t|}{x},
\]

for all \(x > 0\) and \(t \geq 0\).

Exercise 3  (5+10 extra points)

Let \(\{\mathcal{F}_n, n \in \mathbb{N}\}\) be a filtration on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(X_1, X_2, \ldots : \Omega \to \mathbb{R}\) be a sequence of random variables such that \(X_n\) is measurable with respect to \(\mathcal{F}_n\) for each \(n \in \mathbb{N}\). Then, \(\{X_n, n \in \mathbb{N}\}\) is said to be a discrete-time martingale if \(\mathbb{E}(X_{n+1} | X_n) = X_n\). A discrete stopping time with respect to \(\{\mathcal{F}_n, n \in \mathbb{N}\}\) is a random variable \(T : \Omega \to \mathbb{N} \cup \{\infty\}\) such that \(\{T \leq n\} \in \mathcal{F}_n\) for each \(n \in \mathbb{N} \cup \{\infty\}\) where \(\mathcal{F}_\infty = \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n)\).

a) Let \(g : [0, \infty) \to [0, \infty)\) be a monotonously increasing function with \(\lim_{x \to \infty} g(x)/x = \infty\). Show that \(\{X_n, n \in \mathbb{N}\}\) is uniformly integrable if \(\sup_{n \in \mathbb{N}} \mathbb{E}g(|X_n|) < \infty\).

b) Let \(\{X_n, n \in \mathbb{N}\}\) be a discrete-time martingale and let \(T\) be a discrete stopping time such that \(\mathbb{E}|X_T| < \infty\) and \(\lim_{n \to \infty} \mathbb{E}(|X_n|\mathbb{I}(T > n)) = 0\). Show that the sequence of random variables \(X_{\min\{1,T\}}, X_{\min\{2,T\}}, \ldots\) is uniformly integrable.