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Winter term 2014/15

Stochastics II
Exercise Sheet 1

Deadline: October 22, 2014 at 4pm before the exercises

Exercise 1 (3 + 1)

Let $\{N_t, t \geq 0\}$ be a renewal process with i.i.d. interarrival times $T_1, T_2, \ldots$, where $T_1 \sim \text{Exp}(\lambda)$ for some $\lambda > 0$.

a) Show that N_t is Poisson distributed for each $t > 0$.

b) Find the parameter of the Poisson distribution in part (a) for arbitrary $t > 0$.

Exercise 2 (2 + 4)

Let $\{N_t, t \geq 0\}$ be a renewal process with i.i.d. interarrival times $T_1, T_2, \ldots$, where $T_1 \sim \text{U}(1, 1 + 2\theta)$ for an unknown parameter $\theta > 0$. Consider the family of estimators $\{\widehat{\theta}_t, t > 0\}$ for θ with

$$\widehat{\theta}_t = \frac{t - N_t}{N_t}.$$

a) Show that the following holds with probability 1: $\lim_{t \to \infty} \widehat{\theta}_t = \theta$.

b) Determine a symmetric, asymptotic 95%-confidence interval for N_t.

Exercise 3 (3 + 3)

Let $T_1, T_2, \ldots : \Omega \to [0, \infty)$ be a sequence of non-negative and i.i.d. random variables $\mathbb{E}T_1 = \mu \in (0, \infty)$. Let $U_1, U_2, \ldots : \Omega \to \mathbb{R}$ be a sequence of i.i.d. random variables with $\mathbb{E}|U_1| < \infty$. Define the stochastic process $\{X_t, t \geq 0\}$ by

$$X_t = \sum_{k=1}^{\infty} U_k \, \mathbb{I}(T_1 + \ldots + T_k \leq t)$$

for all $t \geq 0$. Show that the following statements hold with probability 1:

a) The sum in the definition of X_t converges for each $t \geq 0$ and

b)
$$\lim_{t \to \infty} \frac{X_t}{t} = \frac{\mathbb{E}U_1}{\mu}.$$

Exercise 4 (1 + 3)

Let $X, Y : \Omega \to [0, \infty)$ be non-negative and independent random variables with distribution functions F_X and F_Y. Moreover, the distribution function of the random variable $X + Y$ is denoted by F_{X+Y}.

a) Show $F_{X+Y}(t) \leq F_X(t) F_Y(t)$ for each $0 < t < \infty$.

b) Show that continuity of F_X implies continuity of F_{X+Y}.