Stochastics II
Exercise Sheet 1

Deadline: October 22, 2014 at 4pm before the exercises

Exercise 1  (3 + 1)
Let \( \{N_t, t \geq 0\} \) be a renewal process with i.i.d. interarrival times \( T_1, T_2, \ldots \), where \( T_1 \sim \text{Exp}(\lambda) \) for some \( \lambda > 0 \).

a) Show that \( N_t \) is Poisson distributed for each \( t > 0 \).

b) Find the parameter of the Poisson distribution in part (a) for arbitrary \( t > 0 \).

Exercise 2  (2 + 4)
Let \( \{N_t, t \geq 0\} \) be a renewal process with i.i.d. interarrival times \( T_1, T_2, \ldots \), where \( T_1 \sim \text{U}(1, 1 + 2\theta) \) for an unknown parameter \( \theta > 0 \). Consider the family of estimators \( \{\hat{\theta}_t, t > 0\} \) for \( \theta \) with
\[
\hat{\theta}_t = \frac{t - N_t}{N_t}.
\]

a) Show that the following holds with probability 1: \( \lim_{t \to \infty} \hat{\theta}_t = \theta \).

b) Determine a symmetric, asymptotic 95%-confidence interval for \( N_t \).

Exercise 3  (3 + 3)
Let \( T_1, T_2, \ldots : \Omega \to [0, \infty) \) be a sequence of non-negative and i.i.d. random variables \( \mathbb{E}T_1 = \mu \in (0, \infty) \). Let \( U_1, U_2, \ldots : \Omega \to \mathbb{R} \) be a sequence of i.i.d. random variables with \( \mathbb{E}|U_1| < \infty \). Define the stochastic process \( \{X_t, t \geq 0\} \) by
\[
X_t = \sum_{k=1}^{\infty} U_k \mathbb{I}(T_1 + \ldots + T_k \leq t)
\]
for all \( t \geq 0 \). Show that the following statements hold with probability 1:

a) The sum in the definition of \( X_t \) converges for each \( t \geq 0 \) and
b) \[
\lim_{t \to \infty} \frac{X_t}{t} = \frac{\mathbb{E}U_1}{\mu}.
\]
Exercise 4  (1 + 3)

Let $X, Y : \Omega \to [0, \infty)$ be non-negative and independent random variables with distribution functions $F_X$ and $F_Y$. Moreover, the distribution function of the random variable $X + Y$ is denoted by $F_{X+Y}$.

a) Show $F_{X+Y}(t) \leq F_X(t)F_Y(t)$ for each $0 < t < \infty$.

b) Show that continuity of $F_X$ implies continuity of $F_{X+Y}$. 