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Winter term 2014/15

Stochastics II
Exercise Sheet 2

Deadline: October 29, 2014 at 4pm before the exercises

Exercise 1 (4)

Let $\{N_t, t \geq 0\}$ be a renewal process with i.i.d. interarrival times $T_1, T_2, \ldots$ and with renewal function $H(t)$. Let $F(t)$ be the distribution function of T_1. Show that it holds

$$F(t) \leq H(t) \leq \frac{F(t)}{1-F(t)}$$

for each t with $F(t) < 1$.

Exercise 2 (3 + 3 + 3)

Let $\{N_t, t \geq 0\}$ be a renewal process . The distribution function of T_1 is denoted by $F(x)$ and the renewal function is denoted by $H(t)$.

a) Show that $\mathbb{E}N_t^2 = \sum_{n=1}^{\infty}(2n-1)F^{*(n)}(t)$ for each $t \geq 0$.

b) Compute the Laplace-Stieltjes transform of $\sum_{n=1}^{\infty} nF^{*(n)}(t)$.

c) Show that $\mathbb{E}N_t^2 = H(t) + 2(H * H)(t)$ for each $t \geq 0$.

Hint: Let $\{F_n : n \in \mathbb{N}\}$ be a sequence of monotonous and right-continuous functions, which are constant on $(-\infty, 0)$. Let g be a non-negative Borel-measurable function. Then it holds:

$$\int_0^{\infty} g(t)\mathrm{d}\left(\sum_{n=1}^{\infty} F_n(t)\right) = \sum_{n=1}^{\infty} \int_0^{\infty} g(t)\mathrm{d}F_n(t)$$

Exercise 3 (3)

Let $\{N_t, t \geq 0\}$ be a delayed renewal process with jump times $S_1, S_2, \ldots$. Define the excess at time $t \geq 0$ by $T(t) = S_{N_t+1} - t$ and let $\mathbb{P}(T(t) \leq x)$ not depend on t. Show that then, the distribution of increments $N_{t_1+t} - N_{t_0+t}, \ldots, N_{t_n+t} - N_{t_{n-1}+t}$ does also not depend on t, for all $0 = t_0 \leq t_1 < \ldots < t_n < \infty$.

Exercise 4 (2 + 1 + 3)

Let $\{N_t, t \geq 0\}$ be a delayed renewal process with holding times $T_1, T_2, \ldots$, where $T_2 \sim \mathrm{U}(0,1)$.

a) Determine the distribution of T_1 such that $\{N_t, t \geq 0\}$ has stationary increments.

b) Let $U \sim \mathrm{U}(0,1)$. Compute the distribution function of the random variable $Z = 1 - \sqrt{1-U}$.

c) Write an R-program in order to simulate $\{N_t, 0 \leq t \leq 100\}$, where T_1 is distributed such that $\{N_t, t \geq 0\}$ has stationary increments. Hand in your R-code and a plot of one realization.