Exercise2

We show that the process $\{N_t, t \geq 0\}$ has stationary and independent increments. Let $0 = t_0 \leq t_1 \leq \ldots \leq t_n < t_{n+1} = \infty$ and $m_1, \ldots m_n$ be arbitrary. Then we get by conditioning on X:

$$\begin{aligned}
&\mathbb{P}(N_{t_1} - N_{t_0} = m_1, \ldots, N_{t_n} - N_{t_{n-1}} = m_n)\\
&\qquad = \sum_{i=0}^{n+1} \int_{t_i}^{t_{i+1}} \mathbb{P}(N_{t_1} - N_{t_0} = m_1, \ldots, N_{t_n} - N_{t_{n-1}} = m_n \mid X = x) \mathrm{d}F_X(x)\\
&\qquad = \sum_{i=0}^{n+1} \int_{t_i}^{t_{i+1}} \mathbb{P}\Big(\{N^{(1)}_{t_j} - N^{(1)}_{t_{j-1}} = m_j, \forall 1 \leq j \leq i\}\\
&\qquad \cap \{N^{(1)}_x + N^{(2)}_{t_{i+1}-x} - N^{(1)}_{t_i} = m_{i+1}\}\\
&\qquad \cap \{N^{(1)}_x + N^{(2)}_{t_{j+1}-x} - N^{(1)}_x - N^{(2)}_{t_j-x} = m_{j+1}, \forall i < j < n\}\Big) \mathrm{d}F_X(x)
\end{aligned}$$

Then, the three event in the last integral are independent from each other because the Poisson processes $N^{(1)}$ and $N^{(2)}$ are independent from each other and Poisson processes have independent increments. Moreover, it holds that $N^{(1)}_x + N^{(2)}_{t_{i+1}-x} - N^{(1)}_{t_i} = N^{(1)}_{x-t_i} + N^{(2)}_{t_{i+1}-x} \sim$ Poi$(\lambda(t_{i+1} - t_i))$, where λ was the intensity of $N^{(1)}$ and $N^{(2)}$. This leads to

$$\begin{aligned}
&\mathbb{P}(N_{t_1} - N_{t_0} = m_1, \ldots, N_{t_n} - N_{t_{n-1}} = m_n)\\
&\qquad = \sum_{i=0}^{n+1} \int_{t_i}^{t_{i+1}} \prod_{j=1}^{n} \mathbb{P}(N_{t_j} - N_{t_{j-1}} = m_j) \mathrm{d}F_X(x)\\
&\qquad = \prod_{j=1}^{n} \mathbb{P}(N_{t_j} - N_{t_{j-1}} = m_j)\\
&\qquad = \prod_{j=1}^{n} \frac{((\lambda(t_j - t_{j-1}))^{m_j}}{m_j!} \exp(-\lambda(t_j - t_{j-1})).
\end{aligned}$$

Now we know that the process N_t has independent increments. Let $h > 0$ be arbitrary. Then we get

$$\begin{aligned}
&\mathbb{P}(N_{t_1+h} - N_{t_0+h} = m_1, \ldots, N_{t_n+h} - N_{t_{n-1}+h} = m_n)\\
&\qquad = \sum_{m_0=0}^{\infty} \mathbb{P}(N_{t_0+h} - N_{t_0} = m_0, N_{t_1+h} - N_{t_0+h} \quad = m_1, \ldots, N_{t_n+h} - N_{t_{n-1}+h} = m_n)
\end{aligned}$$

Now we are in the same case as above and obtain:

$$\begin{aligned}
&= \sum_{m_0=0}^{\infty} \frac{(\lambda h)^{m_0}}{m_0!} \exp(-\lambda h) \prod_{j=1}^{n} \frac{((\lambda(t_j - t_{j-1}))^{m_j}}{m_j!} \exp(-\lambda(t_j - t_{j-1}))\\
&= \prod_{j=1}^{n} \frac{((\lambda(t_j - t_{j-1}))^{m_j}}{m_j!} \exp(-\lambda(t_j - t_{j-1}))
\end{aligned}$$

This shows us that N_t has stationary increments.