Exercise 2

We show that the process \( \{N_t, t \geq 0\} \) has stationary and independent increments. Let \( 0 = t_0 \leq t_1 \leq \ldots \leq t_n < t_{n+1} = \infty \) and \( m_1, \ldots, m_n \) be arbitrary. Then we get by conditioning on \( X \):

\[
\mathbb{P}(N_{t_1} - N_{t_0} = m_1, \ldots, N_{t_n} - N_{t_{n-1}} = m_n)
= \sum_{i=0}^{n+1} \int_{t_i}^{t_{i+1}} \mathbb{P}(N_{t_{j+1}} - N_{t_j} = m_j) \, dF_X(x)
= \sum_{i=0}^{n+1} \int_{t_i}^{t_{i+1}} \mathbb{P}(\{N_{t_j} - N_{t_{j-1}} = m_j, \forall j \leq i\}
\cap \{N_{t_0}^{(1)} + N_{t_{i+1} - x}^{(2)} - N_{t_i}^{(2)} = m_{i+1}\}
\cap \{N_{t_0}^{(1)} + N_{t_{j+1} - x}^{(2)} - N_{t_j}^{(2)} - N_{t_j} = m_{j+1}, \forall i < j < n\}) \, dF_X(x)
\]

Then, the three events in the last integral are independent from each other because the Poisson processes \( N^{(1)} \) and \( N^{(2)} \) are independent from each other and Poisson processes have independent increments. Moreover, it holds that \( N_{t_{i+1} - x}^{(2)} - N_{t_i}^{(1)} = N_{t_{i+1} - t_i}^{(2)} \sim \text{Poi}(\lambda(t_{i+1} - t_i)) \), where \( \lambda \) was the intensity of \( N^{(1)} \) and \( N^{(2)} \). This leads to

\[
\mathbb{P}(N_{t_1} - N_{t_0} = m_1, \ldots, N_{t_n} - N_{t_{n-1}} = m_n)
= \sum_{i=0}^{n+1} \int_{t_i}^{t_{i+1}} \prod_{j=1}^{n} \mathbb{P}(N_{t_j} - N_{t_{j-1}} = m_j) \, dF_X(x)
= \prod_{j=1}^{n} \mathbb{P}(N_{t_j} - N_{t_{j-1}} = m_j)
= \prod_{j=1}^{n} \frac{(\lambda(t_j - t_{j-1}))^{m_j}}{m_j!} \exp(-\lambda(t_j - t_{j-1})).
\]

Now we know that the process \( N_t \) has independent increments. Let \( h > 0 \) be arbitrary. Then we get

\[
\mathbb{P}(N_{t_1+} - N_{t_0+} = m_1, \ldots, N_{t_n+} - N_{t_{n-1}+} = m_n)
= \sum_{m_0=0}^{\infty} \mathbb{P}(N_{t_0+} - N_{t_0} = m_0, N_{t_1+} - N_{t_0+} = m_1, \ldots, N_{t_n+} - N_{t_{n-1}+} = m_n)
\]

Now we are in the same case as above and obtain:

\[
= \sum_{m_0=0}^{\infty} \frac{(\lambda h)^{m_0}}{m_0!} \exp(-\lambda h) \prod_{j=1}^{n} \frac{(\lambda(t_j - t_{j-1}))^{m_j}}{m_j!} \exp(-\lambda(t_j - t_{j-1}))
= \prod_{j=1}^{n} \frac{(\lambda(t_j - t_{j-1}))^{m_j}}{m_j!} \exp(-\lambda(t_j - t_{j-1}))
\]

This shows us that \( N_t \) has stationary increments.