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Stochastics II
Exercise Sheet 3

Deadline: November 5, 2014 at 4pm before the exercises

Exercise 1 (4)

Let $X : \Omega \to [0, \infty)$ be a non-negative random variable with distribution function F and expectation $\mathbb{E}X = \mu \in (0, \infty)$. Let F be differentiable on $(0, \infty)$ and define $F^s(x) = \mu^{-1} \int_0^\infty (1 - F(s))\mathrm{d}s$ for each $x \geq 0$. Then it holds $F^s(x) = F(x)$ for each $x \geq 0$ if and only if $F(x) = (1 - \exp(-x/\mu))\mathbb{I}(x \geq 0)$.

Exercise 2 (4)

Let $\{N_t^{(1)}, t \geq 0\}$ and $\{N_t^{(2)}, t \geq 0\}$ be two independent homogeneous Poisson processes with intensity $0 < \lambda < \infty$. Let $X : \Omega \to [0, \infty)$ be a non-negative random variable, which is independent of $\{N_t^{(1)}, t \geq 0\}$ and $\{N_t^{(2)}, t \geq 0\}$. Define the stochastic process $\{N_t, t \geq 0\}$ by

$$N_t = N_t^{(1)} \mathbb{I}(t \leq X) + \left(N_X^{(1)} + N_{t-X}^{(2)} \right) \mathbb{I}(t > X),$$

for each $t \geq 0$. Show that $\{N_t, t \geq 0\}$ is a Poisson process with intensity λ.

Exercise 3 (2 + 2 + 3)

Let $\{N_t, t \geq 0\}$ be a homogeneous Poisson process with intensity $\lambda > 0$. Let Z be a random variable that is independent of $\{N_t, t \geq 0\}$ with $\mathbb{P}(Z = 1) = 1 - \mathbb{P}(Z = -1) = 1/4$. Define the stochastic process $\{X_t, t \geq 0\}$ by $X_t = Z(-1)^{N_t}$, for each $t \geq 0$.

a) Let $t \geq 0$ arbitrary. Show that $\mathbb{P}(\{N_t \text{ is even}\}) = (1 + \exp(-2\lambda t))/2$.

b) Determine the distribution of X_t for each $t \geq 0$.

c) Compute $\mathbb{E}X_t, \operatorname{Var} X_t$ and $\operatorname{Cov}(X_s, X_t)$ for all $s, t \geq 0$.

Exercise 4 (2 + 3 + 3)

Let $\{N_t, t \geq 0\}$ be a homogeneous Poisson process with intensity $\lambda > 0$ and jump times $S_1, S_2, \ldots$. Define the stochastic process $\{X_t, t \geq 0\}$ by

$$X_t = \sum_{n=1}^{\infty} f(S_n - t),$$

for each $t \geq 0$ with $f(x) = (2x - x^2)\mathbb{I}(0 \leq x \leq 2)$.

a) Show that the sum in the definition of X_t converges for each $t \geq 0$ with probability 1.

b) Compute $\mathbb{E}X_t$.

c) Write an R-code (or Matlab-code) in order to simulate $\{X_t, 0 \leq t \leq 20\}$. Hand in your R-code and a plot of one realization.