Exercise 1 (4)

Let $E = \{1, \ldots, \ell\}$. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)^T$ be a vector with $0 \leq \alpha_i \leq 1$ for each $i \in E$ and $\sum_{i \in E} \alpha_i = 1$. Let $P = (p_{ij})_{i,j \in E}$ be a stochastic matrix, i.e. $0 \leq p_{ij} \leq 1$ for all $i,j \in E$ and $\sum_{j \in E} p_{ij} = 1$ for each $i \in E$. Moreover, let $Z_0, Z_1, \ldots$ be a sequence of i.i.d. random variables with $Z_0 \sim U([0,1])$. Define the sequence of random variables $X_0, X_1, \ldots$ by

$$X_0 = \sum_{k=1}^\ell k \cdot \mathbb{1} \left( \sum_{i=1}^{k-1} \alpha_i < Z_0 \leq \sum_{i=1}^{k} \alpha_i \right)$$

and

$$X_n = \sum_{k=1}^\ell k \cdot \mathbb{1} \left( \sum_{i=1}^{k-1} p_{X_{n-1},j} < Z_n \leq \sum_{i=1}^{k} p_{X_{n-1},j} \right)$$

for each $n \geq 1$. Show that $\{X_n, n \in \mathbb{N}_0\}$ is a time-discrete, homogeneous Markov chain with initial distribution $\alpha$ and transition matrix $P$.

Exercise 2 (4)

Let $\{X_t, t \geq 0\}$ be a Markov process with finite state space $E$, transition function $\{P(h), h \geq 0\}$ and intensity matrix $Q = (q_{ij})_{i,j \in E}$. Show that $\{P(h), h \geq 0\}$ is irreducible if and only if for all $i,j \in E$ there exist a natural number $n$ and distinct $i = i_1, i_2, \ldots, i_n = j \in E$ such that $q_{i_1i_2} \cdot \ldots \cdot q_{i_{n-1}i_n} > 0$.

Exercise 3 (2+2+2+4)

Let $\lambda, \mu > 0$ and let $\{X_t, t \geq 0\}$ be a Markov process with intensity matrix

$$Q = \begin{pmatrix}
-\lambda & \lambda & 0 & 0 & 0 \\
\mu & -(\lambda + \mu) & \lambda & 0 & 0 \\
0 & \mu & -(\lambda + \mu) & \lambda & 0 \\
0 & 0 & \mu & -(\lambda + \mu) & \lambda \\
0 & 0 & 0 & \mu & -\mu
\end{pmatrix}$$

and initial distribution $\alpha = (\alpha_1, \ldots, \alpha_5)^T$. 
a) Show that the transition function $\{P(h), h \geq 0\}$ of $\{X_t, t \geq 0\}$ is irreducible.

b) Compute the stationary initial distribution $\alpha' = (\alpha'_1, \ldots, \alpha'_5)^T$ of $\{X_t, t \geq 0\}$.

c) Let $\alpha = \alpha'$. Show that $\{X_t, t \geq 0\}$ is stationary.

d) Write an R- or Matlab-code in order to simulate $\{X_t, 0 \leq t \leq 30\}$ with $\lambda = 2\mu = 1$ and initial distribution $\alpha = (1/2, 0, 0, 0, 1/2)^T$. Hand in your code and a plot of one realization.