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Stochastics II
Exercise Sheet 6

Deadline: November 26, 2014 at 4pm before the exercises

Exercise 1 (4)

Let $E=\{1,\ldots,\ell\}$. Let $\alpha=(\alpha_1,\ldots,\alpha_\ell)^\top$ be a vector with $0\le\alpha_i\le 1$ for each $i\in E$ and $\sum_{i\in E}\alpha_i=1$. Let $\mathbf{P}=(p_{ij})_{i,j\in E}$ be a stochastic matrix, i.e. $0\le p_{ij}\le 1$ for all $i,j\in E$ and $\sum_{j\in E}p_{ij}=1$ for each $i\in E$. Moreover, let $Z_0,Z_1,\ldots$ be a sequence of i.i.d. random variables with $Z_0\sim \mathrm{U}([0,1])$. Define the sequence of random variables $X_0,X_1,\ldots$ by

$$X_0=\sum_{k=1}^{\ell} k\cdot \mathbb{I}\left(\sum_{i=1}^{k-1}\alpha_i < Z_0 \le \sum_{i=1}^{k}\alpha_i\right)$$

and

$$X_n=\sum_{k=1}^{\ell} k\cdot \mathbb{I}\left(\sum_{i=1}^{k-1}p_{X_{n-1},j} < Z_n \le \sum_{i=1}^{k}p_{X_{n-1},j}\right)$$

for each $n\ge 1$. Show that $\{X_n, n\in\mathbb{N}_0\}$ is a time-discrete, homogeneous Markov chain with initial distribution α and transition matrix $\mathbf{P}$.

Exercise 2 (4)

Let $\{X_t, t\ge 0\}$ be a Markov process with finite state space E, transition function $\{\mathbf{P}(h), h\ge 0\}$ and intensity matrix $Q=(q_{ij})_{i,j\in E}$. Show that $\{\mathbf{P}(h), h\ge 0\}$ is irreducible if and only if for all $i,j\in E$ there exist a natural number n and distinct $i=i_1,i_2,\ldots,i_n=j\in E$ such that $q_{i_1i_2}\cdot\ldots\cdot q_{i_{n-1}i_n}>0$.

Exercise 3 (2+2+2+4)

Let $\lambda,\mu>0$ and let $\{X_t, t\ge 0\}$ be a Markov process with intensity matrix

$$Q=\begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0\\ \mu & -(\lambda+\mu) & \lambda & 0 & 0\\ 0 & \mu & -(\lambda+\mu) & \lambda & 0\\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda\\ 0 & 0 & 0 & \mu & -\mu \end{pmatrix}$$

and initial distribution $\alpha=(\alpha_1,\ldots,\alpha_5)^\top$.

a) Show that the transition function $\{\mathbf{P}(h), h \geq 0\}$ of $\{X_t, t \geq 0\}$ is irreducible.

b) Compute the stationary initial distribution $\alpha' = (\alpha'_1, \ldots, \alpha'_5)^\top$ of $\{X_t, t \geq 0\}$.

c) Let $\alpha = \alpha'$. Show that $\{X_t, t \geq 0\}$ is stationary.

d) Write an R- or Matlab-code in order to simulate $\{X_t, 0 \leq t \leq 30\}$ with $\lambda = 2\mu = 1$ and initial distribution $\alpha = (1/2, 0, 0, 0, 1/2)^\top$. Hand in your code and a plot of one realization.