ulm university universität uulm

Prof. Dr. Volker Schmidt
Matthias Neumann

Winter term 2014/15

Stochastics II
Exercise Sheet 8

Deadline: December 10, 2014 at 4pm before the exercises

Exercise 1 (3+3+2+4+4)

Let $\mu = (\mu_1, \ldots, \mu_n)^\top \in \mathbb{R}^n$ and $K = (k_{i,j})_{i,j=1,\ldots,n}$ a symmetric and positive definite $n \times n$-matrix. The random vector $Z = (Z_1, \ldots, Z_n)^\top$ is said to be (non-degenerate) multivariate normal distributed with mean vector μ and covariance matrix K if the distribution of Z is absolutely continuous with respect to the n-dimensional Lebesgue measure and its density is given by

$$f_Z(z) = \frac{1}{\sqrt{(2\pi)^n \det(K)}} \exp\left(-\frac{1}{2}(z-\mu)^\top K^{-1}(z-\mu)\right),$$

for all $z \in \mathbb{R}^n$. Then, it holds $\mathbb{E}Z_i = \mu_i$ and $\operatorname{Cov}(Z_i, Z_j) = k_{ij}$ for all $i, j \in \{1, \ldots, n\}$. We write $Z \sim N(\mu, K)$. The characteristic function of Z is given by $\varphi(t) = \exp\left(it^\top \mu - \frac{1}{2} t^\top K t\right)$ for all $t \in \mathbb{R}^n$.

A stochastic process $\{V_t, t \geq 0\}$ is said to be a Gaussian process if the random vector $(V_{t_1}, \ldots, V_{t_n})$ is multivariate normal for each $n \in \mathbb{N}$ and for all $0 \leq t_1, \ldots, t_n < \infty$.

a) Let $Z \sim N(o, K)$, where $o = (0, \ldots, 0) \in \mathbb{R}^n$ and let A be a $m \times n$-matrix with rank $\operatorname{rk}(A) = m \leq n$. Show that $Y = AZ \sim N(o, AKA^\top)$.

b) Let $\{X_t, t \geq 0\}$ be a Wiener process. Define the stochastic processes $\{Y_t, t \geq 0\}$ and $\{U_t, t \geq 0\}$ by $Y_t = \exp(X_t)$ and $U_t = \exp(-t/2) X_{\exp(t)}$. Is $\{Y_t, t \geq 0\}$ respectively $\{U_t, t \geq 0\}$ a Gaussian process?

c) Compute $\mathbb{E}Y_t$ and $\operatorname{Cov}(Y_s, Y_t)$ for all $s, t \geq 0$.

d) Is $\{U_t, t \geq 0\}$ a stationary process?

e) You may assume without proof that $\{U_t, t \geq 0\}$ is a Markov process. Determine its initial distribution $\boldsymbol{\alpha}$ and the transition kernel $\{\mathbf{P}(h, x, B), h \geq 0, x \in \mathbb{R}, B \in \mathcal{B}(\mathbb{R})\}$ and show that it holds

$$\boldsymbol{\alpha}([a,b)) = \int_{\mathbb{R}} \mathbf{P}(h, x, [a,b))\boldsymbol{\alpha}(\mathrm{d}x),$$

for all $h \geq 0, a < b \in \mathbb{R}$.

Exercise 2 (3)

Let $\{X_t, t \geq 0\}$ be a Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $z_1, \ldots, z_n \in \mathbb{R}$

be arbitrary. Then, $Z = \{t \geq 0 : X_t \in \{z_1, \ldots, z_n\}\}$ is a random (closed) subset of $[0, \infty)$. You may assume without proof that $\nu(Z)$ is a well-defined (measurable) random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, where ν denotes the one-dimensional Lebesgue measure. Show $\mathbb{P}(\nu(Z) = 0) = 1$.

Exercise 3 (3)

Let $\{X_t, t \geq 0\}$ a Wiener process. Compute $\mathbb{P}(X_t < X_{t+2} < X_{t+1})$ for each $t \geq 0$.