Stochastics III  
Problem Sheet 1

Deadline: October 29, 2014 at noon, before the practical

Exercise 1  
(3 + 2 points)

a) Let $A \in \mathbb{R}^{m \times n}$ with $rk(A) = r$. Show: If $A$ has representation

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11}$ is nonsingular with $rk(A_{11}) = r$, then

$$A^{-} = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{m \times n}$$

is a generalized inverse of $A$, i.e., $AA^{-}A = A$.

b) Determine the generalized inverse of the matrix $A \in \mathbb{R}^{3 \times 4}$,

$$A = \begin{pmatrix} 4 & 7 & -1 & 2 \\ 1 & 2 & 5 & -1 \\ 7 & 13 & 14 & -1 \end{pmatrix}.$$

Exercise 2  
(1 + 3 points)

Let $X = (X_1, \ldots, X_n)^{\top}$ be an $n$-dimensional random vector, where $E(X_i^2) < \infty$, $1 \leq i \leq n$. Furthermore, let $A \in \mathbb{R}^{m \times n}$. Show that

a) $E(AX) = A \ E(X)$,

b) $\text{Cov}(AX) = A \ \text{Cov}(X) \ A^{\top}$,

where the expectation vector and covariance matrix of a random vector $X$ are $E(X) = (E(X_1), \ldots, E(X_n))^{\top}$ and $\text{Cov}(X) = E((X - E(X))(X - E(X))^{\top}) = (\text{Cov}(X_i, X_j))_{i,j=1,\ldots,n}$.

Exercise 3  
(4 points)

Consider the 2-by-2 matrix

$$A = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}.$$

Calculate a diagonal matrix $D$ and a regular matrix $P$ such that $A = P \ D \ P^{-1}$. 
Exercise 4 (3 points)

Give an example of two random variables \(X\) and \(Y\) which are normally distributed but such that the vector \((X, Y)\) is not multivariate non-degenerate normally distributed. Justify your answer.

Exercise 5 (3 + 2 points)

Let \(X = (X_1, X_2, X_3) \sim N(\mu, K)\) with expectation vector \(\mu = (1, 2, 3)\) and covariance matrix

\[
K = \sigma^2 \begin{pmatrix}
2 & \rho & 0 \\
\rho & 1 & \rho \\
0 & \rho & 2
\end{pmatrix}.
\]

a) Determine the marginal distributions of \(X_2\) and \((X_1, X_3)\).

b) For which \(\rho\) are the random variables \(X_1 + X_2 + X_3\) and \(X_1 - X_2 - X_3\) independent?

Exercise 6 (3 + 2 points)

Let \((X, Y)\) bivariate normally distributed with expectation vector \(\mu = (1, 2)\) and covariance matrix

\[
K = \begin{pmatrix}
5 & 2 \\
2 & 3
\end{pmatrix}.
\]

a) Using \texttt{R}, plot the density \(f_{(X,Y)}\) of the random vector \((X, Y)\).

\textit{Hint:} Use the commands \texttt{dmvnorm} and \texttt{persp}.

b) Calculate the expectation and the variance of the random variable \(2X - Y + 3\) in \texttt{R}.