ulm university universität ulm

Dr. Tim Brereton
Lisa Handl

Winter Term 2014/15

Stochastics III

Problem Sheet 1

Deadline: October 29, 2014 at noon, before the practical

Exercise 1 (3 + 2 points)

a) Let $A \in \mathbb{R}^{n \times m}$ with $rk(A) = r$. Show: If A has representation

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is nonsingular with $rk(A_{11}) = r$, then

$$A^- = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{m \times n}$$

is a generalized inverse of A, i.e., $AA^-A = A$.

b) Determine the generalized inverse of the matrix $A \in \mathbb{R}^{3 \times 4}$,

$$A = \begin{pmatrix} 4 & 7 & -1 & 2 \\ 1 & 2 & 5 & -1 \\ 7 & 13 & 14 & -1 \end{pmatrix}.$$

Exercise 2 (1 + 3 points)

Let $\boldsymbol{X} = (X_1, \ldots, X_n)^\top$ be an n-dimensional random vector, where $\mathbb{E}(X_i^2) < \infty,\ 1 \le i \le n$. Furthermore, let $A \in \mathbb{R}^{m \times n}$. Show that

a) $\mathbb{E}(A\boldsymbol{X}) = A\, \mathbb{E}\boldsymbol{X}$,

b) $\text{Cov}(A\boldsymbol{X}) = A\ \text{Cov}(\boldsymbol{X})\ A^\top$,

where the expectation vector and covariance matrix of a random vector $\boldsymbol{X}$ are $\mathbb{E}(\boldsymbol{X}) = \big(\mathbb{E}(X_1), \ldots, \mathbb{E}(X_n)\big)^\top$ and $\text{Cov}(\boldsymbol{X}) = \mathbb{E}\Big(\big(\boldsymbol{X} - \mathbb{E}(\boldsymbol{X})\big)\big(\boldsymbol{X} - \mathbb{E}(\boldsymbol{X})\big)^\top\Big) = \big(\text{Cov}(X_i, X_j)\big)_{i,j=1,\ldots,n}$.

Exercise 3 (4 points)

Consider the 2-by-2 matrix

$$A = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}.$$

Calculate a diagonal matrix D and a regular matrix P such that $A = P\ D\ P^{-1}$.

Exercise 4 (3 points)

Give an example of two random variables X and Y which are normally distributed but such that the vector $(X, Y)^\top$ is not multivariate non-degenerate normally distributed. Justify your answer.

Exercise 5 (3 + 2 points)

Let $\boldsymbol{X} = (X_1, X_2, X_3)^\top \sim \mathrm{N}(\boldsymbol{\mu}, K)$ with expectation vector $\boldsymbol{\mu} = (1, 2, 3)^\top$ and covariance matrix

$$K = \sigma^2 \begin{pmatrix} 2 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 2 \end{pmatrix} .$$

a) Determine the marginal distributions of X_2 und $(X_1, X_3)^\top$.

b) For which ρ are the random variables $X_1 + X_2 + X_3$ and $X_1 - X_2 - X_3$ independent?

Exercise 6 (3 + 2 points)

Let $(X, Y)^\top$ bivariate normally distributed with expectation vector $\boldsymbol{\mu} = (1, 2)^\top$ and covariance matrix

$$K = \begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} .$$

a) Using **R**, plot the density $f_{(X,Y)}$ of the random vector $(X, Y)^\top$.

   Hint: Use the commands `dmvnorm` and `persp`.

b) Calculate the expectation and the variance of the random variable $2X - Y + 3$ in **R**.