Dr. Tim Brereton
Lisa Handl

Winter Term 2014/15

Stochastics III
Problem Sheet 2

Deadline: November 12, 2014 at noon, before the practical

Exercise 1 (4 + 4 points)

Prove Theorem 1.4 in the lecture notes:

Let $\boldsymbol{Y}$ be an n-dimensional random vector with expectation vector $\boldsymbol{\mu} = \mathbb{E}\boldsymbol{Y}$ and covariance matrix $K = \mathrm{Cov}(\boldsymbol{Y})$, such that $\mathrm{rk}(K) = r$ with $r \leq n$. The random vector $\boldsymbol{Y}$ is normally distributed if and only if the following (equivalent) conditions are fulfilled.

a) The characteristic function $\varphi(\boldsymbol{t}) = \mathbb{E}\exp\Big(\mathrm{i}\sum_{j=1}^n t_j Y_j\Big)$ of $\boldsymbol{Y}$ is given by

$$\varphi(\boldsymbol{t}) = \exp\Big(\mathrm{i}\boldsymbol{t}^\top\boldsymbol{\mu} - \frac{1}{2}\,\boldsymbol{t}^\top K\boldsymbol{t}\Big), \qquad \forall\, \boldsymbol{t} = (t_1,\dots,t_n)^\top \in \mathbb{R}^n\,.$$

b) The linear function $\boldsymbol{c}^\top\boldsymbol{Y}$ of $\boldsymbol{Y}$ for every $\boldsymbol{c} \in \mathbb{R}^n$ with $\boldsymbol{c} \neq \boldsymbol{o}$ is normally distributed with

$$\boldsymbol{c}^\top\boldsymbol{Y} \sim \mathrm{N}(\boldsymbol{c}^\top\boldsymbol{\mu}, \boldsymbol{c}^\top K\boldsymbol{c})\,.$$

Exercise 2 (6 points)

Let $\boldsymbol{Z} = (Z_1,\dots,Z_n)^\top \sim \mathrm{N}(\boldsymbol{o}, K)$ a normally distributed random vector with covariance matrix $K = (k_{ij})_{i,j=1,\dots,n}$. Show that for any $i,j,l,m \in \{1,\dots,n\}$ it holds that

$$\mathbb{E}(Z_i Z_j Z_l) = 0$$

and

$$\mathbb{E}(Z_i Z_j Z_l Z_m) = k_{ij}k_{lm} + k_{il}k_{jm} + k_{jl}k_{im}.$$

Exercise 3 (4 + 4 points)

a) Write your own function in **R** to sample from a multivariate normal distribution using Corollary 1.5. The expectation vector and the (positive definite) covariance matrix should be passed as arguments of the function. You may use `rnorm` and `chol` but not `rmvnorm` or similar.

b) Use your function from a) to sample 10000 times from the 2D normal distribution given in Exercise 6 of Problem Sheet 1 and plot the realizations and a 2D histogram.

Exercise 4 (3 + 4 points)

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors, such that $\boldsymbol{X} \sim \mathrm{N}(\boldsymbol{\mu_X}, K_{XX})$ with $\boldsymbol{\mu_X} = (1,2,3)^\top$ and

$$K_{XX} = \begin{pmatrix} 2 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 2 \end{pmatrix},$$

where $\rho \in (0,1)$. About $\boldsymbol{Y}$ we just know that $\boldsymbol{\mu_Y} = \mathbb{E}\boldsymbol{Y} = (-1,0,2)^\top$ and

$$K_{YX} = \big(\mathrm{Cov}(X_i, Y_j)\big)_{ij} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

a) Find the expectation of the random variable

$$Z = 2X_1Y_1 + 4X_2Y_2 + 3X_3Y_3 + X_1Y_3 + X_3Y_1 - X_2Y_3 - X_3Y_2.$$

b) For which ρ are the random variables $X_1^2 + 2X_2^2 + X_3^2$ and $X_1^2 - X_2^2 + X_3^2 - 2X_1X_3$ uncorrelated?

Exercise 5 (4 points)

In **R**, use your simulation method from Exercise 3 to sample 1000 times from the two random variables in Exercise 4 b) and draw a scatter plot of the values you obtain. Use the ρ from your solution of 4 b) and describe the dependency structure that you observe in the plot.

Hint: If you couldn't solve Exercise 3 or Exercise 4 b), use `rmvnorm` and $\rho = 0.5$.