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Winter Term 2014/15

Stochastics III

Problem Sheet 7

Deadline: February 11, 2015 at noon, before the practical

Exercise 1 (3 points)

Consider a linear model $\boldsymbol{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where X is an $n \times m$ matrix with $rk(X) = r < m$. Let $\boldsymbol{a} \in \mathbb{R}^m$. Show that the function $\boldsymbol{a}^\top \boldsymbol{\beta}$ is an estimable function if and only if $\boldsymbol{a}^\top X^- X = \boldsymbol{a}^\top$, where X^- is a generalized inverse of X.

Exercise 2 (3 + 4 points)

Consider the following linear model:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}.$$

a) Show that $\beta_1 + \frac{1}{3}\beta_2 + \frac{2}{3}\beta_3$ is an estimable function.

b) Determine the BLUE estimator for $\beta_1 + \frac{1}{3}\beta_2 + \frac{2}{3}\beta_3$.

Exercise 3 (4 points)

Consider the linear model $\boldsymbol{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ with $\boldsymbol{Y} \in \mathbb{R}^6$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \beta_4)^\top \in \mathbb{R}^4$, $\mathbb{E}\boldsymbol{\varepsilon} = \boldsymbol{o}$, $\mathbb{E}(\varepsilon_i \varepsilon_j) = \delta_{ij}\sigma^2$, where $\sigma^2 > 0$ and

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Show that the set of all estimable functions is given by

$$\{(a_1 + a_2 + a_3)\beta_1 + a_1\beta_2 + a_2\beta_3 + a_3\beta_4 : a_1, a_2, a_3 \in \mathbb{R}\}.$$

Exercise 4 (4 points)

Suppose chocolate bar sales depend on the two factors *wrapping* and *shelf placement*. There are two types of wrapping (red and blue) and three different shelf placements (top, middle and bottom). The following sales numbers have been recorded in different supermarkets:

	red	blue
top	68 65 63 59 67	59 57 54 56 53
middle	59 60 61 48 63	53 47 48 50 51
bottom	47 39 40 46 45	40 39 35 36 37

Determine unbiased estimators of the parameters μ, $\alpha_{i_1}^{(1)}$, $\alpha_{i_2}^{(2)}$ und $\alpha_{i_1 i_2}$, for $i_1 = 1, 2$ and $i_2 = 1, 2, 3$, assuming the balanced two-factor analysis of variance model is appropriate.

Exercise 5 (3 + 2 + 2 + 3 points)

On the course website you will find the file *medicine.txt*, which is structured as follows.

Time until it takes effect (in h)	Medicine A	Medicine B
0.98	1	0
1.08	1	0
1.55	0	1
...	...	...

The information on medicine use is encoded as follows:

- 0: medicine has not been used
- 1: medicine has been used

Consider a linear model for this data with normally distributed error terms. The time until the medicine takes effect is the response variable.

a) What are the expected times until medicine A and medicine B take effect?

b) Show that the regression coefficients β_1, β_2 and β_3 are not estimable functions.

c) Show that the linear combination $\beta_2 - \beta_3$ is an estimable function.

d) Conduct a test for the hypothesis $\beta_2 - \beta_3 = 0$ with significance level $\alpha = 0.05$.

Hint: You can use `ginv()` in the package `MASS` to compute generalized inverses.

Good luck with all your exams and enjoy your holidays! :)