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Winter Term 2015/16

Methods of Monte Carlo Simulation

Problem Sheet 3

Deadline: November 26, 2015 at 4 pm before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (programming) (3 + 2 + 1 points)

a) Write a Matlab program to sample from the binomial distribution with parameters $n = 10$ and $p = 0.2$ using acceptance rejection and the following proposal distributions.

   (i) The uniform distribution on $\{0, \ldots, 10\}$.

   (ii) The geometric distribution with parameter $q = 0.3$, i.e., $P(X = k) = (1 - q)^k q$.

   Hint: You can sample from Geo(q) by drawing from an exponential distribution with parameter $\lambda = -\log(1 - q)$ and rounding off.

b) Draw at least $N = 10^4$ times from this distribution using (i) and (ii) and plot histograms of the sampled values (relative frequencies), as well as of the desired binomial distribution.

   Hint: You might want to use `N = hist(Y,X)` and `bar` for the histograms.

c) Which of the two proposal distributions should be preferred and why?

Exercise 2 (theory) (2 + 3 + 1 points)

Consider a random variable X with density

$$f(x) = \frac{\alpha\, x_m^\alpha}{x^{\alpha+1}}\, \mathbb{I}\{x \geq x_m\},$$

where $x_m, \alpha > 0$.

a) Calculate the cumulative distribution function of X.

b) Explain how you can sample from the distribution of X conditional on $X \geq 2x_m$, assuming your computer can only generate standard uniformly distributed random numbers. Use a combination of the inverse-transform method and acceptance-rejection.

c) Calculate the acceptance probability of your algorithm.

Exercise 3 (programming) (3 points)

Write a Matlab program to sample uniformly from the torus

$$\left\{(x,y,z)\in\mathbb{R}^3 : \left(R-\sqrt{x^2+y^2}\right)^2 + z^2 \leq r^2\right\},$$

where $R = 5$ and $r = 2$, using acceptance-rejection. Draw at least $N = 10^4$ points, plot them in 3D and estimate the acceptance probability of your algorithm.

Hint: You can use the Matlab function `plot3` for plotting.

Exercise 4 (theory) (3 points)

Find the order of the following functions, i.e., for each $i \in \{1,2,3\}$ find $g_i\colon \mathbb{N} \to \mathbb{R}$, such that $f_i(n) = O(g_i(n))$.

a) $f_1(n) = (35+n)\log n$.

b) $f_2(n) = \mathrm{e}^n + n^2$.

c) $f_3(n) = f_1(n) \times f_2(n)$, where $f_1(n)$ and $f_2(n)$ are as above.

Make sure you perform the necessary calculations (show $|f(x)| \leq Cg(x)$ for $x \geq x_0$).

Exercise 5 (theory) (2 points)

Show that the family of Cauchy distributions with densities

$$f(x;x_0,\gamma) = \frac{1}{\pi\gamma}\left(\frac{\gamma^2}{(x-x_0)^2+\gamma^2}\right),$$

where $x_0 \in \mathbb{R}$ and $\gamma > 0$, is a location-scale family.

Exercise 6 (programming) (2 + 1 points)

a) Implement the Box-Muller algorithm to sample from the standard normal distribution. Plot a histogram of $N = 10^5$ values simulated with your algorithm and add the density of the standard normal distribution to your plot.

b) Using a), draw $N = 10^5$ times from $\mathrm{N}(20, 100)$, i.e., from the normal distribution with mean $\mu = 20$ and variance $\sigma^2 = 100$, and plot a histogram of the values you obtain.