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Winter Term 2015/16

Methods of Monte Carlo Simulation

Problem Sheet 4

Deadline: December 10, 2015 at 4 pm before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (2 + 1 + 3 points)

A hen lays M eggs, where M has the Poisson distribution with parameter $\lambda \in (0, \infty)$. Each egg hatches with probability $p \in (0, 1)$ independently of the other eggs. Let K be the number of chicks. Find

a) $\mathbb{E}(K \mid M)$.

b) $\mathbb{E}(K)$.

c) $\mathbb{E}(M \mid K)$.

Exercise 2 (programming) (3 points)

Consider the setting described in Exercise 1 with $\lambda = 5$ and $p = 0.2$. Write a Matlab program to estimate $\mathbb{E}(M \mid K = 2)$ using acceptance-rejection and a sample size of at least $N = 10^4$. Estimate the standard deviation of your estimator.

Hint: For the moment, use `poissrnd` to generate Poisson distributed random numbers in Matlab. We will talk later about how you can generate them yourself.

Exercise 3 (theory) (2 + 2 points)

Let $X \sim \text{Bin}(n, p)$ with $n \in \mathbb{N}$ and $p \in (0, 1)$.

a) Calculate the probability generating function $G(z)$ and the moment generating function $M(t)$ of X.

b) Using a), calculate the first and second moment of X.

Exercise 4 (theory) (4 points)

Let $X \sim \text{Bin}(n, U)$ with $n \in \mathbb{N}$ and $U \sim U(0, 1)$. Using generating functions, show that $X \sim U(\{0, \ldots, n\})$.

Exercise 5 (programming) (2 + 4 points)

Imagine there is a bar in Ulm which offers a casino evening once a week. Whenever a customer orders a beer, a coin is thrown. If the coin shows tails, the customer has to pay. If it shows heads, the beer is for free.

Solve the following simulation and estimation exercises **assuming that your computer can only generate standard uniformly distributed random numbers**. Use only algorithms from the lecture for which you do not have the code and a sample size of at least $N = 10^5$.

a) Suppose you go to this bar on the casino evening and you use your extremely lucky coin, which is heads with probability $p = 0.9$. Estimate the expected number of free beers you can drink before you have to pay.

b) Now assume that the total number of people at the bar is Poisson distributed with mean $\lambda = 30$. Each of them orders one beer and, since the bartender caught you with your lucky coin, everybody has to use a fair coin now. Simulate the distribution of the number of beers that the bartender gets money for and plot a histogram of the values you obtain (relative frequencies).

Exercise 6 (theory) (3 points)

The negative binomial distribution with parameters $r \in \mathbb{N}$ and $p \in (0, 1)$ describes the number of failures in a sequence of independent Bernoulli trials until the r-th success, where the success probability of each Bernoulli trials is p. It has the following pmf.

$$P(X = i) = \binom{i + r - 1}{i} p^r (1 - p)^i, \quad i \in \{0, 1, 2, ...\}.$$

A natural way to simulate this distribution is to simulate r independent geometric random variables (the version which does not count the success) and to add them up.

Show that this approach works.

Hint: You may use without proof that

$$\sum_{k=n}^{\infty} \binom{k}{n} \frac{1}{z^{k+1}} = \frac{1}{(z - 1)^{n+1}}$$

for $n \in \mathbb{N}$ and $z \in \mathbb{R}, |z| > 1$.

Exercise 7 (programming) (4 points)

Something scandalous happened in Ulm. Initially, there are only 3 witnesses. However, on every successive day each person who knows about the scandal tells it to a random number of people who do not know about it. This random number is negative binomially distributed with parameters $r = 6$ and $p = 0.8$ and is independent of the number of people others tell about the scandal. Each person who hears about the scandal starts telling other people the next day.

Write a Matlab program to estimate the expected number of people who know about the scandal after 5 days using a sample size of at least $N = 10^4$ and **assuming that your computer can only generate standard uniform random numbers.**