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Winter Term 2015/16

Methods of Monte Carlo Simulation

Problem Sheet 5

Deadline: January 14, 2016 at 4 pm before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (3 points)

Let X be a random variable with zero mean and variance σ^2. Using Markov's inequality, show that

$$P(X \geq t) \leq \frac{\sigma^2}{\sigma^2 + t^2} \quad \forall t \geq 0.$$

Exercise 2 (theory) (1 + 3 points)

Let $X \sim \text{Exp}(\lambda)$ with $\lambda > 0$.

a) Provide a bound for $P(|X - \mathbb{E}X| \geq \varepsilon)$ as a function of λ and ε.

b) Calculate the above probability as a function of λ and ε. How big is the difference between the bound and the exact probability for $\lambda = 3$ and $\varepsilon = \frac{1}{2}$?

Exercise 3 (theory) (4 + 3 + 3 points)

Consider a sequence of i.i.d. random variables $Z_1, Z_2, \ldots$ with $P(Z_k = 1) = p$ and $P(Z_k = -1) = 1 - p$ and define

$$X_n = \sum_{k=1}^{n} Z_k \quad \forall n \in \mathbb{N}.$$

Then $\{X_n\}_{n \geq 0}$ is called a random walk with parameter p starting at 0. Now suppose you want to estimate the probability $\ell = P(X_n = n)$ using the standard estimator

$$\hat{\ell} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}\{X_n^{(i)} = n\},$$

where $\{X_n^{(1)}\}_{n \geq 0}, \{X_n^{(2)}\}_{n \geq 0}, \ldots$ are i.i.d. copies of $\{X_n\}_{n \geq 0}$.

a) For $n = 5$, use Chebyshev's inequality to find the minimum sample size so that the error is less than or equal to 10^{-3} with probability 0.99.

b) Say you want to ensure that the error of your estimator is, with probability 0.99, smaller than $0.01 \times \ell$. Using Chebyshev's inequality, find $N_{\min}(n)$, the minimum sample size for an arbitrary n. What is the rate of growth of this sample size as $n \to \infty$ (i.e., what is the order of $N_{\min}(n)$)?

c) If, instead, we chose the sample size $N_{\min}(n)$ using the central limit theorem, what would it be?

Exercise 4 (programming) (3 + 3 + 4 points)

Solve the following programming exercises **without using built-in functions for random number generation (not even `rand`)**.

a) Write a Matlab function `myrand` generating your own standard uniformly distributed pseudo-random numbers using the Wichman-Hill generator with parameters $a_1 = 171, a_2 = 170, a_3 = 172$ and $m_1 = 30269, m_2 = 30307, m_3 = 30323$. Your program should have two parameters m and n to specify the size of the matrix which is returned (like `rand`) and choose seeds based on the current time (for example, using `clock`).

b) Write a Matlab function `mybetarnd` to sample from the beta distribution with parameters $p, q > 1$, $p + q > 2$, which has the density

$$f(x; p, q) = \frac{1}{\mathrm{B}(p, q)} x^{p-1}(1-x)^{q-1}\mathbb{I}\{x \geq 0\}.$$

Here B denotes the beta function (`beta` in Matlab). Your function should use the acceptance-rejection method based on a) and have 4 parameters, two for the size (m and n) and two for the parameters p and q. Make sure you use the optimal scaling constant C.

c) Santa wants to bring gifts to the kids in a small village. However, Rudolph ate the list with the naughty and nice kids in this village, so Santa decided to just throw a random number of gifts into each chimney. If there are n kids in a house, the number of gifts he gives to this house is binomially distributed with parameters n and P. The gift probability P for each house is random (depending on Santa's quickly changing mood) and is independent of the probabilities at other houses. Each P follows a beta distribution with parameters $p = 7$ and $q = 2$. In our particular village, there are five houses with 2, 3, 4, 5 and 6 kids, respectively.

Estimate the expected number of kids who stay without gift and the probability that there is at least one house where the kids get no gifts at all. Use a sample size of at least $N = 10^4$ and estimate the standard errors of your estimators.

Merry Christmas and a happy New Year! Enjoy your holidays! :-)