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Winter Term 2015/16

Methods of Monte Carlo Simulation

Problem Sheet 6

Deadline: January 28, 2016 at 4 pm before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (2 points)

Show that $\mathbb{E}\left(4\,\mathbb{I}\left\{U_1^2+U_2^2\leq 1\right\} \mid U_1\right) = 4\sqrt{1-U_1^2}$ for (U_1,U_2) uniformly distributed on the unit square.

Exercise 2 (programming) (2 + 2 + 1 points)

a) Using Exercise 1, write a Matlab program to estimate π using conditional Monte Carlo with a sample size of at least $N = 10^4$.

b) Write a Matlab program to estimate π using antithetic variables with a sample size of at least $N = 10^4$(, i.e., with 5000 pairs of antithetic variables). Justify the choice of your estimator.

c) Estimate the variance of your estimators. Which one performs better?

Exercise 3 (theory) (1 + 3 + 1 points)

Consider a simple random walk starting at 0, i.e.,

$$X_n = \sum_{i=1}^{n} Z_i \quad \forall n \in \mathbb{N}, \quad \text{with} \quad Z_1, Z_2, \dots \text{ i.i.d. and } P(Z_i = 1) = P(Z_i = -1) = \frac{1}{2}.$$

a) Compute the probability $P(X_{10} > 1)$ by hand.

b) Explain how you could use antithetic variables to estimate $P(X_{10} > 1)$. Show that the correlation of the antithetic variables you choose is actually negative and calculate the factor by which you reduce the variance of the estimator with your approach (compared to crude Monte Carlo).

   Hint: You may use the fact that if X_n is a random walk, then $-X_n$ is a random walk as well.

c) How could antithetic variables be used for estimating $P(X_{10} > 1)$ if X_n was a *biased* random walk, i.e., if $P(Z_i = 1) = p$ and $P(Z_i = -1) = 1 - p$ for $p \neq \frac{1}{2}$?

Exercise 4 (programming) (1 + 2 + 1 points)

Consider the estimation problem described in Exercise 2.

a) Write a Matlab program to estimate $P(X_{10} > 1)$ using crude Monte Carlo with a sample size of $N = 10^4$.

b) Write a Matlab program to estimate $P(X_{10} > 1)$ using your approach from 2.b) with a sample size of $N = 10^4$(, i.e., with 5000 pairs of antithetic variables).

c) Compare your estimators by estimating their variance.

Exercise 5 (theory) (1 + 3 + 1 points)

Suppose you want to estimate the value of the integral

$$\ell = \int_0^{10} \frac{9}{50}x^3 - \frac{7}{2}x^2 + 20x - 22 \, \mathrm{d}x$$

using Monte Carlo integration and you would like to use stratified sampling to reduce the variance of your estimator.

a) Write out two estimators of ℓ based on uniform random numbers, one using standard Monte Carlo and one using stratified sampling.

b) Calculate the optimal sample sizes, N_i, for the second estimator and the strata $(0, 1), [1, 2), [2, 6)$ and $[6, 10)$ if you want to have a total sample size of at least $N = 10^5$.

c) What is the variance of your estimator with this choice of sample sizes?

Exercise 6 (programming) (2 + 2 + 1 points)

Consider the same setting as in Exercise 5.

a) In practice, you could probably not calculate optimal sample sizes by hand. Simulate 1000 random variables as a pilot run and *estimate* the optimal proportion of samples which should be drawn from each stratum in 5.b).

b) Implement your stratified-sampling estimator. Use the proportion of samples you estimated in a) and a total sample size of at least $N = 10^5$. Estimate the variance of your estimator

c) Write another program estimating ℓ using standard Monte Carlo and estimate the variance of your estimator.

Exercise 7 (theory) (3 + 2 + 1 points)

One way of interpreting the control variable approach is that it combines two estimators. Suppose X and Y are two random variables with $\mathbb{E}X = \mathbb{E}Y = \ell$. Then any random variable of the form

$$\hat{\ell}_a = aX + (1 - a)Y, \quad a \in \mathbb{R}$$

is an unbiased estimator of ℓ.

a) Calculate the constant a which minimizes the variance of $\hat{\ell}_a$.

b) Calculate the variance of $\hat{\ell}_a$ with this optimal constant.

c) Why is this variance always lower than or equal to $\text{Var}(X)$ and $\text{Var}(Y)$?