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Winter Term 2015/16

Methods of Monte Carlo Simulation

Problem Sheet 7

Deadline: February 11, 2016 at 4 pm before the exercises

Please email your code to lisa.handl@uni-ulm.de AND hand in a printed copy of the code!

Exercise 1 (theory) (2 points)

Suppose we have a "black box" which on command can generate the value of a gamma random variable with parameters $\alpha = \frac{3}{2}$ and $\lambda = 1$. Explain how we can use this black box to approximate $\mathbb{E}\left((X+1)^{-2}\right)$, where X is an exponential random variable with mean 1. Write out your estimator and show that it is unbiased.

Recall that the Gamma distribution with parameters $\alpha, \lambda > 0$ has the density

$$f(x;\alpha,\lambda) = \frac{\lambda \mathrm{e}^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} \mathbb{I}\{x > 0\}.$$

Exercise 2 (theory) (1 + 1 + 2 points)

Consider a random variable $X \sim \mathrm{Exp}(\lambda)$ with $\lambda \in (0, \infty)$. Estimate $P(X > 10)$ by means of importance sampling. Use exponential tilting, i.e., assume that the importance sampling density $g(x)$ belongs to the following parametric family of distributions:

$$\{g(\cdot;\theta) : \theta \in (-\infty, \lambda)\} \quad \text{with} \quad g(x;\theta) = \mathrm{e}^{\theta x - \kappa(\theta)} f(x) \quad \text{and} \quad \kappa(\theta) = \log\left(\mathbb{E}\mathrm{e}^{\theta X}\right).$$

a) Calculate the probability $P(X > 10)$ by hand and write out the importance sampling density $g(x;\theta)$.

b) Find the zero variance density $g^*(x)$.

c) Find the variance minimization parameter θ_{VM} for this problem (by hand).

Exercise 3 (programming) (1 + 2 + 3)

Consider the setting described in Exercise 2. Write a Matlab program to estimate $P(X > 10)$ for $\lambda = \frac{1}{2}$ using

a) standard Monte Carlo

b) importance sampling with θ_{VM} as calculated in Exercise 2

c) importance sampling, estimating θ_{VM} using an initial sample and compare the estimators by estimating their relative errors. Use a sample size of at least $N = 10^5$ in each case.

Exercise 4 (theory) (2 + 3 points)

Let N be a random variable with values in $\mathbb{N}$ and let $Y_1, Y_2, \ldots$ be i.i.d. random variables which are independent of N. Consider

$$X = \sum_{i=1}^{N} Y_i,$$

i.e., the sum of a random number of random variables.

a) Show that $\mathbb{E}X = \mathbb{E}N \cdot \mathbb{E}Y_1$.

b) Let g_N be the probability generating function of N and let M_Y be the moment generating function of Y_1. Show that the moment generating function of X is

$$M_X(t) = g_N(M_Y(t)).$$

Exercise 5 (theory) (4 points)

Consider a discrete-time stochastic process $\{X_n\}_{n\in\mathbb{N}}$ with values in $\mathbb{Z}$. Which of the following random variables are stopping times?

(i) $\tau_1 = \inf\{n \geq 0 : X_n + X_0 = 5\}$

(ii) $\tau_2 = 10$

(iii) $\tau_3 = \sup\{n \geq 0 : X_n \in \{1, 2, 3\}\}$

(iv) $\tau_4 = \inf\{n \geq 0 : Y_{\lceil n/2 \rceil} \geq 4\}$

Justify your answer.

Exercise 6 (programming) (2 + 3 points)

Consider an incompetent businessman. His company starts off with 10000 € but makes a loss, on average, each day. More precisely, the profit or loss on the ith day is described by a random variable $Y_i \sim \mathrm{N}(-30, 10000)$. If his company can get 11000 € in the bank, he is able to sell his company to a competitor. If his company's bank account drops below 0 €, he goes bankrupt.

a) Write a Matlab program to estimate the probability that the business man sells his company before he goes bankrupt using standard Monte Carlo and a sample size of at least $N = 10^4$. Estimate the standard deviation of your estimator.

b) The event that the business man sells his company before he goes bankrupt happens quite rarely. Write a Matlab program to estimate this probability using importance sampling. Estimate the standard deviation of your estimator and compare it to your results from a).

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