Dr. Tim Brereton Winter Term 2015-16

Methods of Monte Carlo Simulation
Reading Course Practice Questions I

1 Theoretical

Exercise 1

Let $\{N_t^{(1)}\}_{t\geq 0}$ and $\{N_t^{(2)}\}_{t\geq 0}$ be two independent homogeneous Poisson processes with positive intensities λ_1 and λ_2, respectively. Show that the sum of those processes, i.e.,

$$\{N_t^{(3)}\}_{t\geq 0} \quad \text{with} \quad N_t^{(3)} = N_t^{(1)} + N_t^{(2)} \quad \forall t \geq 0$$

is again a homogeneous Poisson process and determine its intensity. **Use the infinitesimal definition of Poisson processes**:

A homogeneous Poisson process with intensity $\lambda \in (0, \infty)$ is an increasing, right-continuous integer-valued process $\{N_t\}_{t\geq 0}$ starting from 0 with

- independent increments,
- $P(N_{t+h} - N_t = 0) = 1 - \lambda h + o(h)$ and
- $P(N_{t+h} - N_t = 1) = \lambda h + o(h)$ for $h \downarrow 0$, uniformly in t.

Exercise 2

Let $\{N_t\}_{t\geq 0}$ be an increasing, integer-valued right-continuous process starting from 0 and let $\lambda \in (0, \infty)$. The arrival times $T_1, T_2, \ldots$ of $\{N_t\}_{t\geq 0}$ are defined as

$$T_0 = 0, \quad T_n = \min\{t \geq 0 : N_t = n\} \quad \forall n \in \mathbb{N}.$$

Moreover, the interarrival times $S_1, S_2, \ldots$ of $\{N_t\}_{t\geq 0}$ are defined as $S_n = T_n - T_{n-1}$.

Show that if $\{N_t\}_{t\geq 0}$ has independent and stationary increments and $N_t \sim \mathrm{Poi}(\lambda t)$, then the interarrival times $S_1, S_2, \ldots$ of $\{N_t\}_{t\geq 0}$ are independent and exponentially distributed with parameter λ.

Exercise 3

A radioactive source emits particles in a Poisson process of rate $\lambda \in (0, \infty)$. A Geiger counter placed near the source records a fraction $p \in (0, 1)$ of the particles emitted. So we could say that for each particle T_n we draw a random variable $U_n \sim \mathrm{U}(0, 1)$ and T_n is only recorded if $U_n \leq p$. The random variables $\{U_n\}_{n\in\mathbb{N}}$ are independent of each other and of $\{T_n\}_{n\in\mathbb{N}}$.

Show that the process of recorded particles is again a Poisson process and determine its rate.

Exercise 4

Suppose you are the janitor of a big building and it is your job to replace the lights whenever it is necessary. The lights in the building fail according to a homogeneous Poisson process with intensity $\lambda = 3$ (time given in months). Due to the austerity budget of the university, you are only allowed to buy very low quality lights. Whenever you put one of those lights in, it immediately breaks with probability 0.2. In this case the light is replaced immediately.

a) Calculate the expected number of lights you need per year.

b) Calculate the variance of the number of lights you need per year.

Hint: Use the tower property of conditional expectation (= the law of total expectation).

Exercise 5

Consider a homogeneous Poisson process $\{N_t\}_{t\geq 0}$ with rate $\lambda = 2$.

a) Calculate the expected value and the variance of N_t.

b) Calculate the probability $P(N_2 \geq 5)$ and find $\mathbb{E}(N_5 \mid N_3 = 2)$.

c) Find $\mathbb{E}(N_4 - N_2 \mid N_3 = 4)$.

Exercise 6

Consider an n-dimensional random vector $\boldsymbol{X} \sim \mathrm{N}(o, I)$ and some matrix $A \in \mathbb{R}^{n\times n}$, where $n \in \mathbb{N}$ and where I is the n-dimensional identity matrix. Show that the random vector $\boldsymbol{Z} = A\boldsymbol{X}$ is also $\mathrm{N}(o, I)$ if and only if $A\,A^\top = I$, i.e., if and only if the matrix A is orthogonal.

Exercise 7

Consider the process $X_t = N_t - \lambda t$, where $\{N_t\}_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda \in (0, \infty)$. Show that $\{X_t\}_{t\geq 0}$ has the following properties:

a) $\mathbb{E}X_t = 0$.

b) $\mathbb{E}(X_t \mid X_s) = X_s$ for all $0 < s < t$.

c) $\mathbb{E}|X_t| < \infty$ for all $t \geq 0$.

2 Practical / Programming

Exercise 8

a) Write a Matlab program to simulate a homogeneous Poisson process with intensity $\lambda = 5$ in the interval $[0, 10]$

1. by generating its interarrival times
2. by using a grid with a mesh size of $h = 0.05$

and plot one of its realizations (for each method).

b) Run your simulation programs from a) repeatedly (10^4 times) and plot a histogram of the values of N_1 you obtain (relative frequencies) next to the pdf of its theoretical distribution for each method.

Exercise 9

a) Write a Matlab program to simulate an inhomogeneous Poisson process $\{N_t\}_{t\geq 0}$ with rate function $\lambda(t) = 3(\cos(t) + 1)$ in the interval $[0, 10]$

1. by thinning a homogeneous Poisson process
2. by using a grid with a mesh size of $h = 0.01$

and plot one of its realizations (for each method). Add the rate function to your plot.

b) Estimate the expected value of $N_{3\pi}$ using a) 1 using a sample size of at least $N = 10^4$. What is the true expected value?

Exercise 10

Suppose you go fishing at some lake and the time (in minutes) that you have to wait until a fish bites follows a Gamma distribution with parameters $k = 6$ and $\theta = 7$. Since you're very talented you never lose a fish that bites.

a) Write a Matlab program to simulate the process $\{N_t\}_{t\geq 0}$ describing the number of fish you caught until time t and estimate how many fish you catch on average if you go fishing for 5 hours. Plot a realization of the process up to 5 hours.

b) Now suppose the weight of the fish (in pounds) is distributed like $\frac{1}{4}Y$, where Y has a χ^2 distribution with 4 degrees of freedom. Let $\{X_t\}_{t\geq 0}$ be the total weight of the fish you caught until time t. Write a Matlab program to simulate $\{X_t\}_{t\geq 0}$. Estimate the probability that you catch more than 6 pounds of fish and plot a realization of the process up to 5 hours.

Exercise 11

Suppose you have an insurance company, which obeys the following model: The insured persons pay a premium at a constant rate c of 4000 € per month (in total, not per person). Claims arrive in a homogeneous Poisson process with rate $\lambda = 25$ and claim sizes are independent and exponentially distributed with parameter $\gamma = 0.005$, where time is given in months and claim sizes are in €. You start off with a capital of 10000 €.

a) Write a Matlab program to simulate the process X_t describing the amount of money you have at time t for a period of 1, 3 and 6 months (i.e., for $t \in [0, 1]$, $t \in [0, 3]$, ...) and plot a realization of it each time.

b) Estimate the probability that you go bankrupt within the first year and estimate the relative error of your estimator. What happens if you increase the premiums such that you get 5000 € per month? Use a sample size of at least $N = 10^4$.