ulm university universität uulm

Prof. Dr. Evgeny Spodarev
Dipl.-Math. Stefan Roth

WS 2015/2016

Stochastics II
Exercise Sheet 10

Due to: Wednesday, 13th of January 2016

Exercise 1 (10 Points)

Let X be an ID random variable[1] with Lévy characteristics (a, b, ν).

(a) Let $c : \mathbb{R} \to \mathbb{R}$ be an arbitrary function with the following asymptotical properties:

$$c(x) = 1 + o(|x|), \quad |x| \to 0 \tag{1}$$
$$c(x) = \mathcal{O}(1/|x|), \quad |x| \to \infty \tag{2}$$

Show that the characteristic function φ of X can be written in the form

$$\varphi(z) = \exp\left[-\frac{1}{2}\tilde{a}z^2 + iz\tilde{b}_c + \int_{\mathbb{R}} \left(e^{izx} - 1 - izxc(x)\right)\nu(dx)\right]$$

where $\tilde{a} \geq 0$ and $\tilde{b}_c \in \mathbb{R}$.

(b) Show that the following functions $\rho, \sigma, \tau : \mathbb{R} \to \mathbb{R}$ fullfill the conditions (1) and (2):

$$\rho(x) = 1/(1 + x^2)$$
$$\sigma(x) = \sin(x)/x$$
$$\tau(x) = \begin{cases} 1 & ; \ |x| \leq 1 \\ 1/|x| & ; \ |x| > 1 \end{cases}$$

(c) Find three more examples (different from ρ, σ, τ) for a function $c : \mathbb{R} \to \mathbb{R}$ which fullfill the conditions (1) and (2).

Exercise 2 (4 Points)

Let φ be a characteristic function. Show that $\psi : \mathbb{R} \to \mathbb{C}$ defined by

$$\psi(z) = \frac{1-b}{1-a} \cdot \frac{1 - a\varphi(z)}{1 - b\varphi(z)},$$

$0 \leq a < b < 1$, is an ID characteristic function.

Exercise 3 (8 Points)

Let $X_1, \ldots, X_n$ be independent and infinitely divisible random variables. Assume that the Lévy measure of X_j is absolutely continuous w.r.t. the Lebesgue measure, for all $j = 1, \ldots, n$. For arbitrary $\mu_1, \ldots, \mu_n \in \mathbb{R}$ define the random variable $Y = \sum_{j=1}^{n} \mu_j X_j$. Show that Y is inifinitely divisible as well and determine its Lévy characteristics.

[1] A random variable is infinitely divisible if and only if its characteristic function admits a Lévy-Khintchine representation.