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WS 2015/2016

Stochastics II
Exercise Sheet 11

Due to: Wednesday, 13th of January 2016

Exercise 1 (5 Points)

Let $\{X(t),\ t \geq 0\}$ be a Gamma process with parameters $b, p > 0$, i.e. $X(t) \sim \Gamma(b, pt)$. Show that $\{X(t),\ t \geq 0\}$ is a subordinator with Laplace exponent

$$\xi(u) = \int_0^\infty (1 - e^{-uy})\nu(dy)$$

where $\nu(dy) = py^{-1}\exp(-by)dy$, $y > 0$. Note that the Laplace exponent of $\{X(t),\ t \geq 0\}$ is defined as the function $\xi : [0, \infty) \to [0, \infty)$ with $E\exp(-uX(t)) = \exp(-t\xi(u))$ for arbitrary $t, u \geq 0$.

Exercise 2 (7 Points)

Let $\{X(t);\ t \geq 0\}$ be a compound Poisson process with Lévy measure

$$\nu(dx) = \frac{\lambda\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx, \quad x \in \mathbb{R},$$

where λ, $\sigma > 0$. Let furthermore $W = \{W(t);\ t \geq 0\}$ be a Wiener process and $N = \{N(t);\ t \geq 0\}$ be a Poisson process with intensity 2λ. Assume that N and W are independent. Show[1] that the process $\{\sigma W(N(t));\ t \geq 0\}$ has the same finite dimensional distributions as X.

Exercise 3 (6 Points)

Let T be a subordinator with density given by

$$f_{T(t)}(s) = \frac{t}{2\sqrt{\pi}} s^{-3/2} \exp(-t^2/4s) \mathbb{1}_{(0,\infty)}(s), \quad s \in \mathbb{R}.$$

Show[2] that T is a 1/2-stable [3] subordinator.

Exercise 4 (4 Points)

Let $W = \{W(t);\ t \geq 0\}$ be a Wiener process and τ be a $\alpha/2$-stable subordinator, $\alpha \in (0, 2)$. Show that $\{W(\tau(s));\ s \geq 0\}$ is an α-stable Lévy process.

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[1] You can use Theorem 5.6.4 without proof.

[2] Find a differential equation by differentiating the Laplace transform of $T(t)$.

[3] A subordinator $X = \{X(t);\ t \geq 0\}$ is called *α-stable* (cf. Example 4.1.2) if $\varphi_{X(1)}(s) = \exp(\int_{\mathbb{R}}(e^{isx} - 1)\nu(dx))$, $s \in \mathbb{R}$, with Lévy measure ν given by

$$\nu(dx) = \begin{cases} \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{x^{1+\alpha}} dx & , x > 0 \\ 0 & , x \leq 0. \end{cases}$$