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WS 2015/2016

Stochastics II
Exercise Sheet 12

Due to: Wednesday, 20th of January 2016

Exercise 1 (10 Points)

Let X be a random variable on some probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{B} \subseteq \mathcal{F}$ be a σ-field. If $E|X| < \infty$ the *conditional expectation* of X w.r.t. $\mathcal{B}$ is defined as the $\mathcal{B}$-measurable random variable Y with the property

$$\int_B Y(w)P(dw) = \int_B X(w)P(dw), \quad \forall B \in \mathcal{B}.$$

Instead of Y we use the notation $E(X|\mathcal{B})$. Such a random variable exists and is a.s. uniquely determined. Show the following properties of $E(X|\mathcal{B})$:

(a) $E(X|\{\emptyset, \Omega\}) = EX$ and $E(X|\mathcal{F}) = X$ a.s.

(b) If $X \leq Z$ a.s. and $E|Z| < \infty$ then $E(X|\mathcal{B}) \leq E(Z|\mathcal{B})$ a.s.

(c) If X is $\mathcal{B}$-measurable and $E|Z|, E|XZ| < \infty$ it holds $E(XZ|\mathcal{B}) = X \cdot E(Z|\mathcal{B})$ a.s.

(d) $E(c|\mathcal{B}) = c$ a.s. for every constant $c \in \mathbb{R}$.

(e) If $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{F}$ it holds

$$E(E(X|\mathcal{B}_2)|\mathcal{B}_1) = E(E(X|\mathcal{B}_1)|\mathcal{B}_2) = E(X|\mathcal{B}_1) \quad a.s.$$

(f) If the σ-fields $X^{-1}(\mathcal{B}_{\mathbb{R}})$ and $\mathcal{B}$ are independent then it holds $E(X|\mathcal{B}) = EX$ a.s.

Exercise 2 (4 Points)

Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with

$$P(X_i = 1) = p_+, \quad P(X_i = -1) = p_-, \quad P(X_i = 0) = p_0, \quad i \in \mathbb{N},$$

where $p_+, p_-, p_0 \in (0,1)$ such that $p_+ + p_- + p_0 = 1$. Define $S_n = \sum_{i=1}^n X_i$, $S_0 = 0$. Show: There exists some $\alpha \neq 0$ such that $\{e^{\alpha S_n} : n \in \mathbb{N}_0\}$ is a martingale w.r.t. the natural filtration if and only if $p_+ \neq p_-$.

Exercise 3 (4 Points)

Let $\{X_n, \ n \in \mathbb{N}\}$ be (independent) coin flippings, i.e. $P(X_n = 1) = p \in [0,1]$, $P(X_n = -1) = 1 - p$. Let $a > 0$ be the initial capital and e_1 the money that we bet before the first flip. For $n \geq 2$ define $e_n = C_{n-1}(X_1, \dots, X_{n-1})$ with some function $C_{n-1} : \{-1, 1\}^{n-1} \to \mathbb{R}_+$. Our financial situation after the n-th flip is modeled by the random variables $S_n = S_{n-1} + X_n \cdot C_{n-1}(X_1, \dots, X_{n-1})$, $n \geq 2$ and $S_1 = a + X_1 \cdot e_1$. Show that $\{S_n, \ n \in \mathbb{N}\}$ is a

$$\begin{cases} \text{supermartingale} & \text{if } p < \frac{1}{2} \\ \text{martingale} & \text{if } p = \frac{1}{2} \\ \text{submartingale} & \text{if } p > \frac{1}{2} \end{cases}$$

w.r.t. the natural filtration $\{\sigma(X_1, \ldots, X_n)\}_{n \in \mathbb{N}}$.

Exercise 4 (5 Points)

For a stopping time τ we define the stopped σ-algebra $\mathcal{F}_\tau$ by

$$\mathcal{F}_\tau := \{B \in \mathcal{F} : \; B \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for arbitrary } t \geq 0\}.$$

Let ρ and γ be stopping times w.r.t. the filtration $\{\mathcal{F}_t, \; t \geq 0\}$. Show the following statements:

(a) $A \cap \{\rho \leq \gamma\} \in \mathcal{F}_\gamma, \; \forall \; A \in \mathcal{F}_\rho$.

(b) $\mathcal{F}_{\min\{\rho,\gamma\}} = \mathcal{F}_\rho \cap \mathcal{F}_\gamma$.

Exercise 5 (4 Points)

Let X and Y be arbitrary random variables on some probability space $(\Omega, \mathcal{F}, P)$. Define $E(Y|X) := E(Y|\sigma(X))$, where $\sigma(X) := \sigma(\{X^{-1}(B), B \in \mathcal{B}(\mathbb{R})\})$. The conditional variance of Y given X is defined by

$$Var(Y|X) := E((Y - E(Y|X))^2|X)$$

Show that

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X)).$$