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WS 2015/2016

Stochastics II
Exercise Sheet 13

Due to: Wednesday, 27th of January 2016

Exercise 1 (3 Points)

Let X and Y be random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}X^2 = \mathbb{E}Y^2 < \infty$ and let $\mathcal{G} \subset \mathcal{F}$ be a σ-field. Show that if $\mathbb{E}(Y|\mathcal{G}) = X$ a.s. then it holds $X = Y$ a.s.

Exercise 2 (3 Points)

Let $\{X_n\}_{n\in\mathbb{N}_0}$ and $\{Y_n\}_{n\in\mathbb{N}_0}$ be submartingales w.r.t. the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$. Prove the following statements:

(a) $\{\max\{X_n, Y_n\}\}_{n\in\mathbb{N}_0}$ is a submartingale w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$.

(b) $\{X_n^2\}_{n\in\mathbb{N}_0}$ is not necessarily a submartingale.

Exercise 3 (5 Points)

Let $\{S_n = a + \sum_{i=1}^n X_i\}_{n\in\mathbb{N}}$ be a symmetric random walk, where $a > 0$ and $P(X_i = -1) = P(X_i = 1) = 1/2$, $i \in \mathbb{N}$. Show that $\{M_n = \sum_{i=0}^n S_i - S_n^3/3\}_{n\in\mathbb{N}}$ is a martingale w.r.t. the natural filtration.

Exercise 4 (6 Points)

(a) Let $g : [0, \infty) \to [0, \infty)$ be a monotonously increasing function such that

$$\frac{g(x)}{x} \to \infty, \quad (x \to \infty).$$

Show that the sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables is uniformly integrable if $\sup_{n\in\mathbb{N}} \mathbb{E}(g(|X_n|)) < \infty$. Is this also necessary for $\{X_n\}_{n\in\mathbb{N}}$ to be uniformly integrable?

(b) Let $X = \{X_n\}_{n\in\mathbb{N}}$ be a martingale and $\tau : \Omega \to \mathbb{N}$ be a finite stopping time. Show that $\{X_{\tau\wedge n}\}_{n\in\mathbb{N}}$ is a martingale, if $\mathbb{E}|X_\tau| < \infty$ and $\lim_{n\to\infty} \mathbb{E}(|X_n| \mathbb{1}_{\{\tau>n\}}) = 0$.

Exercise 5 (5 Points)

Let $X = \{X(t);\ t \geq 0\}$ be a càdlàg submartingale w.r.t. a filtration $\{\mathcal{F}_t;\ t \geq 0\}$ with $\mathbb{E}X(t) = 0$ and $\mathbb{E}(X(t))^2 < \infty$ for all $t \geq 0$. Show[1] that

$$P(\sup_{0\leq s\leq t} X(s) > x) \leq \frac{\mathbb{E}(X(t))^2}{x + \mathbb{E}(X(t))^2},$$

for arbitrary $x > 0$ and $t \geq 0$.

[1]For $y \geq 0$ consider the process $\{(X(t) + y)_+^2;\ t \geq 0\}$ and apply Doob's Theorem. Choose an appropriate y.