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# Stochastics II
# Exercise Sheet 14

Due to: Wednesday, 3rd of February 2016

**Exercise 1** (3 Points)

Let $\xi_1, \xi_2, \dots$ be a sequence of independent random variables with $\mathbb{E}\xi_i = 0$, for all $i \in \mathbb{N}$. Show that for each $k \in \mathbb{N}$ the sequence $\{X_n^{(k)}\}_{n\in\mathbb{N}_0}$ given by

$$X_0^{(k)} = 0, \quad X_n^{(k)} = \sum_{1\le i_1<\dots<i_k\le n} \xi_{i_1}\dots\xi_{i_k}$$

is a martingale w.r.t. the filtration generated by $\xi_1, \xi_2, \dots$.

**Exercise 2** (4 Points)

Let $X = \{X_n\}_{n\in\mathbb{N}}$ be a martingale and $\tau : \Omega \to \mathbb{N}$ be a finite stopping time. Show that $\{X_{\tau\wedge n}\}_{n\in\mathbb{N}}$ is uniformly integrable, if $\mathbb{E}|X_\tau| < \infty$ and $\lim_{n\to\infty} \mathbb{E}(|X_n|\mathbb{1}_{\{\tau>n\}}) = 0$.

**Exercise 3** (7 Points)

Let $X_1, X_2, \dots$ be i.i.d. with $P(X_1 = 1) = P(X_1 = -1) = 1/2$ and

$$S_n = \sum_{k=1}^{n} X_k, \quad n \in \mathbb{N}.$$

Define $T = \inf\{n : \ |S_n| > \sqrt{n}\}$ and $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ , $n \in \mathbb{N}$.

(a) Show that $T$ is a stopping time w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.

(b) Show[1] that $\{G_n\}_{n\in\mathbb{N}}$ with $G_n = S^2_{T\wedge n} - T \wedge n$ is a martingale w.r.t. $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$.

(c) Show that $|G_n| \le 4T$ for all $n \in \mathbb{N}$.

**Exercise 4** (6 Points)

Let $W = \{W(t);\ t \ge 0\}$ be a Wiener process. For $a > 0$ denote by

$$\tau_a = \inf\{t \ge 0;\ |W(t)| = a\},$$

the first exit time of $W$ from the strip $(-a, a)$.

(a) Show that $P(W(\tau_a) = a) = P(W(\tau_a) = -a) = 1/2$.

(b) Compute $\mathbb{E}e^{-\lambda\tau_a}$ for $\lambda > 0$.

(c) Compute[2] $\mathbb{E}\tau_a^2$.

[1] See Exercise 4, (b) on exercise sheet 13.
[2] Use part (b).