Prof. Dr. Evgeny Spodarev
Dipl.-Math. Stefan Roth

WS 2015/2016

Stochastics II
Exercise Sheet 15

Due to: Wednesday, 10th of February 2016

Exercise 1 (5 Points)

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, such that $S_n := \sum_{k=1}^n X_k$ converges almost surely for $n \to \infty$. Furthermore let $\{a_n\}_{n\in\mathbb{N}}$ be a monotonously increasing sequence of non-negative real numbers with $a_n \stackrel{n\to\infty}{\longrightarrow} \infty$. Show[1] that

$$\frac{1}{a_n}\sum_{k=1}^n a_k X_k \longrightarrow 0, \quad a.s.$$

as $n \to \infty$.

Exercise 2 (3 Points)

Let X be a non-negative random variable on some probability space $(\Omega, \mathcal{F}, P)$ and $T : \Omega \to \Omega$ a measure preserving map. Show[2] that

$$\sum_{k=1}^{\infty} X(T^k(\omega)) = \infty,$$

for almost all $\omega \in \Omega$ with $X(\omega) > 0$.

Exercise 3 (4 Points)

Let X be a non-negative random variable on some probability space $(\Omega, \mathcal{F}, P)$ and $T : \Omega \to \Omega$ a measure preserving map. Show[3] that $E(X) = E(X \circ T)$, i.e.

$$\int_\Omega X(T(\omega))P(d\omega) = \int_\Omega X(\omega)P(d\omega).$$

Exercise 4 (4 Points)

Let $(\Omega, \mathcal{F}, P) = ([0,1), \mathcal{B}([0,1)), \nu)$, where ν denotes the Lebesgue measure on $[0,1)$. Let $\lambda \in (0,1)$.

(a) Show that $T(x) = (x+\lambda) \pmod 1$ is a measure preserving map, where $a \pmod b := a - \left\lfloor \frac{a}{b} \right\rfloor \cdot b$ for $a \in \mathbb{R}$ and $b \in \mathbb{Z}$.

(b) Show that $T(x) = \lambda x$ and $T(x) = x^2$ are not measure preserving.

Note: We will provide one more exercise till Tuesday.

[1] It holds $\frac{1}{a_n}\sum_{k=1}^n a_k X_k = \frac{1}{a_n}\sum_{k=1}^n (a_k - a_{k-1})(S_n - S_{k-1})$, $a_0 := 0$, $S_0 := 0$.
[2] Use Poincaré's Theorem (Th. 6.2.1).
[3] Use algebraic induction.