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Stochastics II
Exercise Sheet 1

Due to: Wednesday, 21st of October 2015

In the following we use the notation from the lecture notes. For a random function $X = \{X(t),\ t \in T\}$ associated with $(\mathcal{S}_t, \mathcal{B}_t)_{t\in T}$ let $\mathcal{S}_{t_1,\dots,t_n} = \mathcal{S}_{t_1} \times \cdots \times \mathcal{S}_{t_n}$ as well as $\mathcal{B}_{t_1,\dots,t_n} = \mathcal{B}_{t_1} \otimes \cdots \otimes \mathcal{B}_{t_n}$ where $n \in \mathbb{N}$ and $t_1, \dots, t_n \in T$. All random elements are defined on a common probability space $(\Omega, \mathcal{A}, P)$.

Exercise 1 (3 + 4 + 4 Points)

Let $\mu \in \mathbb{R}^n$ and $K \in \mathbb{R}^{n\times n}$ a positive semi-definite, symmetric matrix. A random vector $X = (X_1, \dots, X_n)^\top$ is called gaussian with expectation μ and covariance matrix K, if its characteristic function[1] ψ_X is given by

$$\psi_X(z) = \mathbb{E}\exp\left(iz^\top X\right) = \exp\left(iz^\top\mu - \frac{1}{2}z^\top K z\right), \quad z \in \mathbb{R}^n,$$

and we write $X \sim N(\mu, K)$. Show the following properties of X:

(a) For any matrix $A \in \mathbb{R}^{m\times n}$ and any $\nu \in \mathbb{R}^m$ it holds $AX + \nu \sim N(A\mu + \nu, AKA^\top)$.

(b) For arbitrary $1 \le i_1 < i_2 < \cdots < i_l \le n$, $l \in \{1, \dots, n\}$ the random vector $\tilde{X} = (X_{i_1}, \dots, X_{i_l})^\top$ is gaussian. Specify its expectation and covariance matrix.

(c) The components of X are independent, if and only if the covariance matrix K is diagonal, i.e. for $K = (k_{ij})_{i,j=1,\dots,n}$ it holds $k_{ij} = 0$ for $i \neq j$.

Exercise 2 (4 Points)

Show the existence of a gaussian random function and specify the measurable spaces $(\mathcal{S}_{t_1,\dots,t_n}, \mathcal{B}_{t_1,\dots,t_n})$.

Hint: Combine Exercise 1.8.1 from the lecture notes (without proof) and Exercise 1 above.

Exercise 3 (4 Points)

Let $X = \{X(t),\ t \in T\}$ and $Y = \{Y(t),\ t \in T\}$ be two stochastic processes on a common probability space $(\Omega, \mathcal{A}, P)$ with values in $(\mathcal{S}, \mathcal{B})$.

(a) Show that if X and Y are stochastically equivalent then they have the same distribution, i.e. $P_X = P_Y$.

(b) Give an example of two stochastic processes which are not equivalent but have the same distribution.

[1] The distribution of a random vector is (as in the case of random variables) uniquely determined by its characteristic function.

Exercise 4 (5 Points)

Let $X = \{X(t),\ t \in T\}$ be a random function. Show that the vector $(X(t_1), \dots, X(t_n))^\top$ is $\mathcal{A}|\mathcal{B}_{t_1,\dots,t_n}$-measurable for every $n \in \mathbb{N}$ and arbitrary $t_1, \dots, t_n \in T$.

Hint: It holds $(X(t_1, \omega), \dots, X(t_n, \omega))^\top = p_{t_1,\dots,t_n} \circ X(\omega)$ with the projection map $p_{t_1,\dots,t_n} : \{x,\ x(t) \in \mathcal{S}_t,\ t \in T\} \to \mathcal{S}_{t_1,\dots,t_n},\ x \mapsto (x(t_1), \dots, x(t_n))^\top$.