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WS 2015/2016

Stochastics II
Exercise Sheet 2

Due to: Wednesday, 28th of October 2015

Exercise 1 (4 Points)

Let $(T, |\cdot|_T)$, $(\mathcal{S}, |\cdot|_\mathcal{S})$ be normed spaces and $X = \{X(t),\ t \in T\}$ a random function on T with values in $\mathcal{S}$. Let furthermore $K \subset T$ be a compact subset of T. Prove that if X is stochastically continuous on K then it is also uniformly stochastically continuous, i.e. $\forall\ \varepsilon, \eta > 0\ \exists\ \delta > 0$ such that for $s, t \in K$ with $|s - t|_T < \delta$ it holds that $P(|X(t) - X(s)|_\mathcal{S} > \varepsilon) < \eta$.

Exercise 2 (4 Points)

Give examples for a stochastic process $X = \{X(t),\ t \in T\}$ with the following properties (with proof!).

(a) X is not separable[1].

(b) X is stochastically continuous but not L_1-continuous.
Hint: Find a process which grows large on a contracting interval.

Exercise 3 (4 Points)

Consider a stochastic process $X = \{X(t),\ t \in [0,1]\}$ which consists of independent and identically distributed random variables with density $f(x)$, $x \in \mathbb{R}$. Show that such a process can not be stochastically continuous in $t \in [0,1]$.

Exercise 4 (4 Points)

Show that the Poisson process is stochastically continuous although it does not possess any a.s. continuous modifications.

Exercise 5 (3 Points)

Let $\mu : T \to \mathbb{R}$ be a measurable function and $K : T \times T \to \mathbb{R}$ be a positiv semi-definite symmetric function. Prove that a random function $X = \{X(t);\ t \in T\}$ exists with $\mathbb{E}X(t) = \mu(t)$ and $\mathsf{cov}(X(s), X(t)) = K(s,t)$, $s, t \in T$.

[1] X is called separable, if there exists a countable and dense subset $D \subset T$ and a fixed event A with $P(A) = 0$, such that

$$\{\omega \in \Omega;\ X(t) \in B\ \forall\ t \in I \cap D\} \backslash \{\omega \in \Omega;\ X(t) \in B\ \forall\ t \in I\} \subset A$$

for any closed subset $B \subset \mathbb{R}$ and open $I \subset T$.

Exercise 6 (4 Points)

(a) Show: In general the condition in Theorem 1.3.1 does not imply the existence of a continuous modification if condition (1.3.1) holds only for $\delta = 0$.

(b) The real-valued gaussian process $X = \{X(t);\ t \geq 0\}$ with $X(t) = e^{-t}W_{e^{2t}},\ t \geq 0$ is called Ornstein-Uhlenbeck process, where $W = \{W(t);\ t \geq 0\}$ is the Wiener process (i.e. a centered gaussian process with covariance function $C_W(s,t) = \min\{s,t\},\ s,t \geq 0$). Show that the Ornstein-Uhlenbeck process has a continuous version.