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Stochastics II
Exercise Sheet 4

Due to: Wednesday, 11th of November 2015

Exercise 1 (2 Points)

Prove Remark 2.1.4: For a delayed renewal-process $N = \{N(t);\ t \geq 0\}$ with delay T_1 it holds

(a) $H(t) = \sum\limits_{k=0}^{\infty} \left(F_{T_1} * F_{T_2}^{*k}\right)(t)$, $t \geq 0$.

(b) $\hat{l}_H(s) = \frac{\hat{l}_{T_1}(s)}{1-\hat{l}_{T_2}(s)}$, $s > 0$.

Exercise 2 (2 Points)

Let $N = \{N(t);\ t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Find

(a) $P(N_1 = 2, N_2 = 3, N_3 = 5)$.

(b) $P(N_1 \leq 2, N_2 = 3, N_3 \geq 5)$.

(c) the probability that $N(t)$ is even respectively odd, $t > 0$.

Exercise 3 (4 Points)

Let $N = \{N(t);\ t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Calculate

$$P(N(s) = i | N(t) = n)$$

for $0 < s < t$ and $i = 1, \ldots, n$.

Exercise 4 (6 Points)

Let $N = \{N(t);\ t \geq 0\}$ be a Poisson process with intensity $\lambda > 0$. Let Y be a random variable with $P(Y = 1) = P(Y = -1) = 1/2$ which is independent of the process N. Define a stochastic process $X = \{X(t);\ t \geq 0\}$ by $X(t) = Y \cdot (-1)^{N(t)}$.

(a) Let $t > 0$ be arbitrary but fixed. Calculate the probability that $X(t)$ is 1 (-1, respectively).

(b) Calculate the covariance function of X.

Exercise 5 (8 Points)

Let $\mathcal{B}_0(\mathbb{R}^d) := \{B \in \mathcal{B}(\mathbb{R}^d),\ \nu_d(B) < \infty\}$, where ν_d denotes the d-dimensional Lebesgue-measure. Let furthermore $\mu : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ be a locally finite measure (i.e. $\mu(B) < \infty$ for every $B \in \mathcal{B}_0(\mathbb{R}^d)$). The non-homogeneous Poisson process can be defined as $N = \{N(B),\ B \in \mathcal{B}(\mathbb{R}^d)\}$ with the following properties:

1.) $N(B) \sim Poi(\mu(B))$ for every $B \in \mathcal{B}_0(\mathbb{R}^d)$.

2.) $N(B_1), \dots, N(B_n)$ are independent random variables for pairwise disjoint $B_i \in \mathcal{B}_0(\mathbb{R}^d)$, $i = 1, \dots, n$ and arbitrary $n \in \mathbb{N}$.

μ is called the intensity measure of N.

(a) Let $A_1,\ A_2 \in \mathcal{B}_0(\mathbb{R}^d)$ be arbitrary. Show that for fixed $k \in \mathbb{N}_0$ and $l = 0, \dots, k$, the probability $P(N(A_1 \cup A_2) = k, N(A_1 \cap A_2) = l)$ is given by

$$\frac{\mu^k(A_1 \cup A_2)}{k!} e^{-\mu(A_1 \cup A_2)} \binom{k}{l} \left(\frac{\mu(A_1 \cap A_2)}{\mu(A_1 \cup A_2)}\right)^l \left(1 - \frac{\mu(A_1 \cap A_2)}{\mu(A_1 \cup A_2)}\right)^{k-l}.$$

(b) Let $n \in \mathbb{N}$, $k_1, \dots, k_n \in \mathbb{N}_0$ and $B_1, ..., B_n \in \mathcal{B}_0(\mathbb{R}^d)$ pairwise disjoint. Verify that for $k = \sum_{i=1}^n k_i$ and $B = \cup_{i=1}^n B_i$ it holds

$$P(N(B_1) = k_1, \dots, N(B_n) = k_n | N(B) = k) = \frac{k!}{k_1! \dots k_n!} \frac{\mu^{k_1}(B_1) \dots \mu^{k_n}(B_n)}{\mu^k(B)}$$

provided that $\mu(B) > 0$.