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WS 2015/2016

Stochastics II
Exercise Sheet 5

Due to: Wednesday, 18th of November 2015

Exercise 1 (4 Points)

Let $N^{(1)} = \{N^{(1)}(t);\ t \geq 0\}$ and $N^{(2)} = \{N^{(2)}(t);\ t \geq 0\}$ be two independent Poisson processes with intensities $\lambda_1, \lambda_2 > 0$, i.e. the sequences $\{T_k^{(1)}\}_{k\in\mathbb{N}}$ and $\{T_k^{(2)}\}_{k\in\mathbb{N}}$ of interarrival times are independent. Show that $N = \{N(t);\ t \geq 0\}$ defined by

$$N(t) = N^{(1)}(t) + N^{(2)}(t), \quad t \geq 0$$

is a Poisson process with intensity $\lambda_1 + \lambda_2$.

Exercise 2 (5 Points)

Let $N = \{N(t),\ t \geq 0\}$ be a renewal process. Define $\chi(t) = S_{N(t)+1} - t$ and $C(t) = t - S_{N(t)}$, $t > 0$. χ is called **excess time**, C **current life time**. Now let N be a Poisson process with intesity $\lambda > 0$.

(a) Calculate the distribution of $\chi(t)$.

(b) Show that the distribution of the current life time is given by

$$P(C(t) \leq s) = \exp(-\lambda t)\delta_t(s) + \int\limits_0^{\min\{s,t\}} \lambda \exp(-\lambda x) dx, \quad s \in [0, t],$$

with $\delta_t(s) = 1$, if $s = t$ and 0 otherwise.

Exercise 3 (5 Points)

Let $N = \{N(t),\ t \geq 0\}$ be a homogeneous Poisson process with intensity $\lambda > 0$ and let $U = \{U_i\}_{i\in\mathbb{N}}$ be an i.i.d. sequence of non-negative random variables defined on the same probability space as N. Let furthermore N and U be independent. Define the process $X = \{X(t); t \geq 0\}$ by

$$X(t) = \sum_{i=1}^{N(t)} U_i, \quad t \geq 0.$$

Show that

(a) the Laplace transform of $X(t)$ is given by

$$\hat{l}_{X(t)}(s) = m_{N(t)}(\hat{l}_{U_1}(s)), \quad s \geq 0,$$

where $m_{N(t)}(s) = \mathbb{E}s^{N(t)}$ denotes the generating function of $N(t)$.

(b) if $U_1 \sim Exp(\gamma)$, $\gamma > 0$, then $\hat{l}_{X(t)}(s) = \exp(-\frac{\lambda t s}{\gamma + s})$, $s \geq 0$.

Exercise 4 (6 Points)

Let $\{T_k\}_{k \in \mathbb{N}}$ be a sequence of i.i.d. random variables with $T_1 \sim Exp(\lambda)$, $\lambda > 0$. For $n \in \mathbb{N}$, define $S_n = \sum_{k=1}^{n} T_k$. Show that for any $n \in \mathbb{N}$ it holds that

$$\left(\frac{S_1}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}} \right) \stackrel{d}{=} (U_{(1)}, \ldots, U_{(n)}),$$

where $U_1, \ldots, U_n$ are i.i.d. random variables with $U_1 \sim U([0,1])$ and $U_{(k)}$ denotes the k-th order statistic of $(U_1, \ldots, U_n)$.

Exercise 5 (4 Points)

Consider a Poisson counting measure N as in Exercise 5, Sheet 4. Let $B_0 \in \mathcal{B}(\mathbb{R}^d)$ and $\tilde{\mu}(\cdot) := \mu(\cdot \cap B_0)$. Show, that the random counting measure $\tilde{N} = \{\tilde{N}(B),\ B \in \mathcal{B}(\mathbb{R}^d)\}$ defined by $\tilde{N}(\cdot) := N(\cdot \cap B_0)$ is a Poisson counting measure with intensity measure $\tilde{\mu}$.