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WS 2015/2016

Stochastics II
Exercise Sheet 6

Due to: Wednesday, 25th of November 2015

Exercise 1 (6 Points)

Let $X = \{X(t),\ t \geq 0\}$, $X(t) = \sum_{j=1}^{N(t)} U_j$, $t \geq 0$, be a compound Poisson process with parameters $\lambda,\ F_U$.

(a) Determine its characteristic function $\varphi_{X(t)}(z) = \mathbb{E}e^{izX(t)}$, $z \in \mathbb{R}$, $t \geq 0$.

(b) Let $\mathbb{E}|U_1| < \infty$. Show that

$$\varphi_U'(z) = \frac{1}{\lambda t} \frac{\varphi_{X(t)}'(z)}{\varphi_{X(t)}(z)}, \quad z \in \mathbb{R},$$

where φ_U denotes the characteristic function of U_1.

Exercise 2 (9 Points)

Let $N = \{N(t),\ t \in [0, \infty)]\}$ be a Cox process with random intensity measure $\Lambda = \{\Lambda(B),\ B \in \mathcal{B}(\mathbb{R}_+)\}$. Show the following statements:

(a) If the moment generating function $M_{\Lambda((0,t])}$ of $\Lambda((0,t])$ exists in a neighborhood U of 0, then

$$M_{N(t)}(s) = M_{\Lambda((0,t])}(e^s - 1), \quad s \in \tilde{U},$$

where $M_{N(t)}(s)$ denotes the moment generating function of $N(t)$, $t > 0$ and $\tilde{U} = \{s \in \mathbb{R},\ e^s - 1 \in U\}$.

(b) If $\mathbb{E}|\Lambda((0,t])| < \infty$, then $\mathbb{E}N(t) = \mathbb{E}\Lambda((0,t])$.

(c) If $\mathbb{E}\Lambda^2((0,t]) < \infty$, then $\mathsf{Var}N(t) = \mathbb{E}\Lambda((0,t]) + \mathsf{Var}\Lambda((0,t])$.

Exercise 3 (3 Points)

Give an algorithm of how to simulate the trajectories of a Wiener process $W = \{W(t),\ t \in [0,1]\}$ by using the independence and the distribution of the increments of W.

Exercise 4 (4 Points)

An approximation for the Wiener process $W = \{W(t),\ t \in [0,1]\}$ is given by

$$W_n(t) = \sum_{k=1}^{n} S_k(t) Z_k \qquad (1)$$

where S_k are the Schauder functions, $t \in [0,1]$, $k \geq 1$ and $Z_k \overset{\text{i.i.d.}}{\sim} N(0,1)$. The convergence of the sequence in (1) ($n \to \infty$) has to be understood in L^2-sense for all $t \in [0,1]$. Show[1] that

$$W_n(t) \xrightarrow{L^2} W(t) \quad (n \to \infty)$$

[1]You can use that the L^2-limit of $\{W_n(t)\}_{n\in\mathbb{N}}$ is a Wiener process without proof (see also Theorem 3.2.1.).