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Stochastics II
Exercise Sheet 7

Due to: Wednesday, 2nd of December 2015

Exercise 1 (4 Points)

Let $W_1 = \{W_1(t),\ t \geq 0\}$ and $W_2 = \{W_2(t),\ t \geq 0\}$ be two independent Wiener processes. For which numbers $a, b \in \mathbb{R}$ is the process $W = \{W(t),\ t \geq 0\}$ defined by $W(t) = aW_1(t) + bW_2(t)$, $t \geq 0$, a Wiener process as well?

Exercise 2 (4 Points)

Let $W = \{W(t),\ t \in [0, 1]\}$ be a Wiener process. Use the extension of Kolmogorov's continuity criterion[1] to show that for any $\gamma < 1/2$ the process W has a Hölder continuous modification with Hölder exponent γ.

Exercise 3 (4 Points)

Show[2] that the Wiener process $W = \{W(t),\ t \in [0, 1]\}$ is a.s. not absolutely continuous.

Exercise 4 (1 + 4 Points)

Let $W = \{W(t),\ t \in [0, 1]\}$ be a Wiener process and $L = \operatorname{argmax}_{t \in [0,1]} W(t)$. Show[3] that

$$P(L \leq x) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad x \in [0, 1].$$

Exercise 5 (5 + 3 Points)

Write a program in **R** which simulates the trajectory of a Wiener process on $[0, 1]$

(a) by using approximation with Schauder functions and input parameters `t` and `m`, where `t` is a finite dimensional vector of locations in $[0, 1]$ and `m` is the cutt-off parameter of the series expansion.

(b) exactly by using the algorithm from sheet 6, exercise 3 with input parameter `t` defined as in (a).

Simulate and plot a trajectory in both cases. Take `m`$= 12$ in (a) and `t`$= (t_0, \ldots, t_{1000})$ in (b), where $t_0 = 0$ and $t_k = k/1000$, $k = 1, \ldots, 1000$.

[1]**Kolmogorov's continuity criterion:** Let $X = \{X(t),\ t \in [0, 1]\}$ be a real valued stochastic process. Suppose there exist $p > 0$, $c > 0$, $\varepsilon > 0$ so that for every $s, t \geq 0$,

$$\mathbb{E}|X(t) - X(s)|^p \leq c|t - s|^{1+\varepsilon}.$$

Then, there exists a modification $\tilde{X}$ of X which is a.s. continuous, and even α-Hölder continuous for any $\alpha \in (0, \varepsilon/p)$.

[2]Use that any absolutely continuous function is a.e. differentiable.

[3]Use the second statement from Theorem 3.3.2 as well as the fact that $\max_{r \in [0,t]} W(r) \stackrel{d}{=} |W(t)|$.