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WS 2015/2016

Stochastics II
Exercise Sheet 8

Due to: Wednesday, 9th of December 2015

Exercise 1 (4 Points)

Let $W = \{W(t);\ t \geq 0\}$ be a Wiener process and define $Z = \{t \geq 0;\ W(t) = 0\}$. Show that

$$P(\nu(Z) = 0) = 1,$$

where ν denotes the Lebesgue measure on $\mathbb{R}$.

Exercise 2 (8 Points)

Let $W = \{W(t);\ t \in [0,1]\}$ be a Wiener process. Consider again the approximations of W from exercise 5, sheet 7. Therefore let $0 = t_0 < t_1 < \cdots < t_m = 1$ and $W_m^{(1)}(t)$ be the approximation obtained from exercise 3, sheet 6 by interpolating $W(t_0), \ldots, W(t_m)$ linearly. Let furthermore $W_n^{(2)}(t)$ be the approximation by Schauder functions[1]. Calculate in both cases the L^2-error of the approximation defined by

$$e(W_n^{(i)}, W) = \left(\mathbb{E} \left[\int_0^1 |W_n^{(i)}(t) - W(t)|^2 dt \right] \right)^{1/2}, \quad i = 1, 2.$$

Let $n \geq 2$. How should $m = m(n)$ at least be chosen such that $e(W_m^{(1)}, W) \leq e(W_n^{(2)}, W)$?

Exercise 3 (8 Points)

Let $W = \{W(t);\ t \geq 0\}$ be a Wiener process. Define the process of the maximum as $M = \{M(t) = \max_{s \in [0,t]} W(s);\ t \geq 0\}$. Show[2]:

(a) The probability density of $M(t)$ is given by

$$f_{M(t)}(x) = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{x^2}{2t}\right) \mathbb{1}\{x \geq 0\}, \quad x \in \mathbb{R}.$$

(b) The expectation and the variance of $M(t)$ are given by

$$\mathbb{E}M(t) = \sqrt{\frac{2t}{\pi}}, \qquad \mathsf{Var}M(t) = t\left(1 - \frac{2}{\pi}\right).$$

(c) Let $\tau(x) = \min\{s \in \mathbb{R};\ W(s) = x\}$ be the first time when W attains the value x. Calculate the density[3] of $\tau(x)$ and show that $\mathbb{E}\tau(x) = \infty$.

[1] Consider $n = 2^{k+1}$ for some $k \in \mathbb{N}_0$.
[2] In part (a) use the fact that $P(M(t) > x) = 2P(W(t) > x)$.
[3] For $x < 0$ use that $-W \overset{d}{=} W$.

Exercise 4 (6 Points)

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with $X_1 \sim N(0,1)$. Show[4] that

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log n}} = 1 \quad \text{a.s.}$$

[4] You can use without proof that for $x > 0$ it holds that

$$\frac{1}{x+\frac{1}{x}} e^{-x^2/2} \leq P(X_1 \geq x) \leq \frac{1}{x} e^{-x^2/2}.$$