Prof. Dr. Evgeny Spodarev
Dipl.-Math. Stefan Roth

WS 2015/2016

Stochastics II
Exercise Sheet 9

Due to: Wednesday, 16th of December 2015

Exercise 1 (5 Points)

A certain class of infinitely divisible random variables are the so called stable random variables. A usual definition is given as follows: A random variable X is called stable if there are parameters $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$, and $\mu \in \mathbb{R}$ such that its characteristic function has the following form:

$$\mathbb{E} \exp(izX) = \begin{cases} \exp\left\{-\sigma^\alpha |z|^\alpha \left(1 - i\beta \operatorname{sign}(z) \tan \frac{\pi\alpha}{2}\right) + i\mu z\right\} & , \quad \text{if } \alpha \neq 1 \\ \exp\left\{-\sigma |z| \left(1 + i\beta \frac{2}{\pi} \operatorname{sign}(z) \log |z|\right) + i\mu z\right\} & , \quad \text{if } \alpha = 1 \end{cases},$$

$z \in \mathbb{R}$, where

$$\operatorname{sign}(z) = \begin{cases} 1 & , \quad \text{if } z > 0 \\ 0 & , \quad \text{if } z = 0 \\ -1 & , \quad \text{if } z < 0. \end{cases}$$

Show: If X is stable then for any $n \geq 2$ there is a positive number a_n and a real number b_n such that

$$X_1 + \cdots + X_n \stackrel{d}{=} a_n X + b_n,$$

where $X_1, \ldots, X_n$ are independent copies of X.

Exercise 2 (4 Points)

Show[1] that the function $\varphi : \mathbb{R} \to \mathbb{C}$, given by

$$\varphi(z) = \exp(\psi(z)), \qquad \psi(z) = 2 \sum_{k=-\infty}^{\infty} 2^{-k} (\cos(2^k z) - 1)$$

is the characteristic function of an infinitely divisible random variable.

Exercise 3 (6 Points)

Let X be a random variable with characteristic function φ.

(a) Show[2]: If X is infinitely divisible $\Rightarrow$ $\varphi(z) \neq 0, \forall\, z \in \mathbb{R}$

(b) Find an example for a random variable which is not infinitely divisible (with proof).

Exercise 4 (4 Points)

Let $P(s) = \sum_{k=0}^{\infty} p_k s^k$ where $p_k \geq 0$, $p_0 > 0$ and $\sum_{k=0}^{\infty} p_k = 1$. Assume that $\log(\frac{P(s)}{P(0)})$ is a power series with positive coefficients. If φ is the characteristic function of an arbitrary distribution F show[3] that $P(\varphi)$ is the characteristic function of an infinitely divisible random variable.

[1]Use the Lévy-Kchinchin representation, find a Lévy measure for φ

[2]Let $\varphi(z) = (\varphi_n(z))^n$. Show that $\lim_{n\to\infty} |\varphi_n(z)|^2$ is cont. on $\mathbb{R}$. Use the fact that $|\varphi_n|^2$ is a characteristic function.

[3]Find the Lévy measure in terms of F^{*n}.