Stochastic Geometry
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SS 2016

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1. Exercise sheet
Deadline: November, 4th, 12:15

Exercise 1: Is the origin in the convex hull of random points?
(4 Credits)

Let $K \subseteq \mathbb{R}^2$ be a convex and compact set with $0 \in \operatorname{int} K$. Let $X_1, \ldots, X_n$ be independent random points distributed uniformly in K. Show that there is a constant $c < 1$ (depending on K) with

$$\mathbb{P}(0 \notin \operatorname{conv}\{X_1, \ldots, X_n\}) \in O(c^n), \quad n \to \infty.$$

Exercise 2: The line $g(\varphi, p)$
(2+4=6 Credits)

Here we want to examine the line $g(\varphi, p)$ introduced in the lecture (in order to prepare an example we will treat on November, 2nd). Recall that $g(\varphi, p)$ is for $\varphi \in [0, 2\pi)$ and $p > 0$ the line whose normal vector pointing away from 0 forms an angle of φ with the first unit vector (measured counter-clockwise from the first unit vector to the normal vector) and which has distance p from the origin.

a) If $g(\varphi, p)$ is not parallel to the second unit vector, it is the graph of an affine function. Determine this function.

Now let $K = [-\frac{a}{2}, \frac{a}{2}]^2$ for $a > 0$ be the axis-parallel square centered at the origin of side-length $a > 0$. Assume from now on $0 < \varphi < \pi/4$.

b) Show that $g(\varphi, p)$ intersects K if and only if $p \leq \frac{a}{2}(\sin\varphi + \cos\varphi)$. Show that the intersection points of $g(\varphi, p)$ with the boundary of K are

$$\begin{cases} (\frac{p}{\cos\varphi} - \frac{a}{2}\tan\varphi, \frac{a}{2}) \text{ and } (\frac{p}{\cos\varphi} + \frac{a}{2}\tan\varphi, -\frac{a}{2}) & \text{if } p \leq \frac{a}{2}(\cos\varphi - \sin\varphi) \\ (\frac{p}{\cos\varphi} - \frac{a}{2}\tan\varphi, \frac{a}{2}) \text{ and } (\frac{a}{2}, \frac{p}{\sin\varphi} - \frac{a}{2\tan\varphi}) & \text{if } p \in [\frac{a}{2}(\cos\varphi - \sin\varphi), \frac{a}{2}(\cos\varphi + \sin\varphi)]. \end{cases}$$

Exercise 3: The number of edges using R
(2+4+1+1+3+1=12 Credits)

Again, let $X_1, \ldots, X_n$ be independent random points distributed uniformly in some convex and compact set $K \subseteq \mathbb{R}^2$ with interior points. We want to examine the number of vertices - or equivalent the number of faces - of $\operatorname{conv}\{X_1, \ldots, X_n\}$.

a) What can you tell (from the results of the lecture) about the asymptotic behavoir of the expected value of the number of faces when K is a square and when K is a ball.

b) Write a function that determines for points $x_1, \ldots, x_n \subseteq \mathbb{R}^2$ the number of faces of $\operatorname{conv}\{x_1, \ldots, x_n\}$. *Hint: Implement the indicator ϵ_{ij} from the proof of the lemma in the lecture. For this, use that a segment from a point x_i to a point x_j is an edge of $\operatorname{conv}\{x_1, \ldots, x_n\}$ if and only if i and j are minimizers or maximizers of the map $k \mapsto \langle x_k, n_{ij}\rangle$, where n_{ij} is a vector perpendicular to the line segment from x_i to x_j.*

c) Apply the function from part b) to 100 points $X_1, \ldots, X_{100}$ chosen uniformly from the unit square $K = [0, 1]^2$.

d) Repeat part c) 1000 times. Use the results to estimate mean and variance of the number of faces and plot a histogram.

e) Repeat part c) and d) for the unit ball

$$K := B_1(0) := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

Hint: In order to simulate a random point distributed uniformly in the unit ball proceed as follows: Simulate a point X distributed uniformly in the square $[-1, 1]^2$. If $X \in B_1(0)$, you are done. Otherwise repeat this until you get a point in $B_1(0)$.

f) Compare the results from part d) and e) in the view of a).