Exercise sheet 12 (total – 16 points) till January 25, 2016

Exercise 12-1 (5 points)

Let X, Y be arbitrary random variables on $(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{E}|X| < \infty$, $\mathbf{E}|Y| < \infty$ $\mathbf{E}|XY| < \infty$ and let $\mathcal{G} \subset \mathcal{F}$ be an arbitrary sub-σ-Algebra of $\mathcal{F}$. Then it holds a.s. that

1. $\mathbf{E}(X|\{\emptyset, \Omega\}) = \mathbf{E}X$, $\mathbf{E}(X|\mathcal{F}) = X$,
2. $\mathbf{E}(aX + bY|\mathcal{G}) = a\mathbf{E}(X|\mathcal{G}) + b\mathbf{E}(Y|\mathcal{G})$ for arbitrary $a, b \in \mathbb{R}$.
3. $\mathbf{E}(X|\mathcal{G}) \le \mathbf{E}(Y|\mathcal{G})$, if $X \le Y$,
4. $\mathbf{E}(XY|\mathcal{G}) = Y\mathbf{E}(X|\mathcal{G})$, if Y is a $(\mathcal{G}, \mathcal{B}(\mathbb{R}))-$measurable random variable,
5. $\mathbf{E}(\mathbf{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbf{E}(X|\mathcal{G}_1)$, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are sub-σ-algebras of $\mathcal{F}$ with $\mathcal{G}_1 \subset \mathcal{G}_2$.

Exercise 12-2 (4 points)

Let X and Y be random variables on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The conditional variance is defined by $\mathbf{Var}(Y|X) = \mathbf{E}((Y - \mathbf{E}(Y|X))^2|X)$. Show that:

1. $\mathbf{E}Y = \mathbf{E}(\mathbf{E}(Y|X))$,
2. $\mathbf{Var}Y = \mathbf{E}(\mathbf{Var}(Y|X)) + \mathbf{Var}(\mathbf{E}(Y|X))$.

Exercise 12-3 (4 points)

For a stopping time τ define the stopped σ-algebra $\mathcal{F}_\tau$ as follows:

$$\mathcal{F}_\tau = \{B \in \mathcal{F} : B \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for arbitrary } t \ge 0\}.$$

Let now σ and τ be stopping times w.r.t. the filtration $\{\mathcal{F}_t, t \ge 0\}$. Show:

(a) $A \cap \{\sigma \le \tau\} \in \mathcal{F}_\tau \; \forall A \in \mathcal{F}_\sigma$.

(b) $\mathcal{F}_{\min\{\sigma,\tau\}} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

Exercise 12-4 (3 points)

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent $N(0,1)-$distributed random variables. Let $S_n = \xi_1 + \cdots + \xi_n$. Prove that the sequence $X_n, n \ge 1$ given by

$$X_n = \frac{1}{\sqrt{n+1}} \exp\left(\frac{S_n^2}{2(n+1)}\right)$$

is a martingale w.r.t. the filtration $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n), n \ge 1$.