Exercise sheet 13 (total – 24 points) till February 8, 2017

Exercise 13-1 (3 points)

Let $\mathcal{F}_n, n \geq 0$ be a filtration on $(\Omega, \mathcal{F}, \mathbf{P})$. Let σ and τ be two stopping times w.r.t. $\mathcal{F}_n, n \geq 0$ such that $\sigma(\omega) \leq \tau(\omega), \omega \in \Omega$ and $A_n = \{\omega : \sigma(\omega) < n \leq \tau(\omega)\}$. Show that A_n is $\mathcal{F}_{n-1}$-measurable for each $n \geq 1$.

Exercise 13-2 (3 points)

Let a random sequence $\{X_n, n \geq 0\}$ be a submartingale w.r.t. the filtration $\{\mathcal{F}_n, n \geq 0\}$. Prove that there exists a non-negative martingale $\{M_n, n \geq 0\}$ such that

$$\max(0, X_n) \leq M_n, n \geq 0, \quad \text{and} \quad \sup_{n\geq 0} \mathbf{E}[\max(0, X_n)] = \sup_{n\geq 0} \mathbf{E} M_n.$$

Exercise 13-3 (4 points)

(*Doob decomposition*) Let a random sequence $\{X_n, n \geq 0\}$ be a submartingale w.r.t. the filtration $\{\mathcal{F}_n, n \geq 0\}$. Prove that there exists a martingale $\{M_n, n \geq 0\}$ and non-decreasing integrable random sequence $\{A_n, n \geq 0\}$ such that $A_0 = 0$, A_n is $\mathcal{F}_{n-1}$-measurable for each $n \geq 1$, and

$$X_n = M_n + A_n, \forall n \geq 0.$$

Exercise 13-4 (3 points)

(*Polya's Urn Scheme*) An urn contains r red and g green balls. At each time, we draw a ball out, then put it back, and add c more balls of the color drawn. Let X_n be the fraction of green balls after the n^{th} drawing. Prove that $X_n, n \geq 1$ is a martingale.

Exercise 13-5 (5 points)

(L_p maximum inequality.) If $X_n, \geq 0$ is a martingale then for any $1 < p < \infty$

$$\mathbf{E}\left(\max_{0\leq n\leq N} |X_n|\right)^p \leq \left(\frac{p}{p-1}\right)^p \mathbf{E}(|X_N|^p).$$

Hint: Firstly, consider $(\max_{0\leq n\leq N} X_n) \wedge M$ for some constant $M > 0$. Then apply Doob's inequality for it.

Exercise 13-6 (6 points)

(a) Prove that if $X_n, n \geq 0$ is a non-negative supermartingale, then $\exists \lim_{n\to\infty} X_n = X$ a.s. and $\mathbf{E}X \leq \mathbf{E}X_0$.
Hint: Use upcrossing inequality: if $Y_n, n \geq 0$ is a submartingale then $(b-a)\mathbf{E}U_n \leq \mathbf{E}(Y_n - a)_+ - \mathbf{E}(Y_0 - a)_+$, where U_n is the number of upcrossings of interval (a, b) by $Y_m, m \geq 0$ completed by time n.

(b) Let random sequences $X_n, n \geq 0$ and $Y_n, n \geq 0$ be a.s. non-negative, integrable and adapted to the filtration $\mathcal{F}_n, n \geq 0$. Suppose $\mathbf{E}(X_{n+1}|\mathcal{F}_n) \leq (1 + Y_n)X_n$, $\forall n \geq 0$ with $\sum_{n=0}^{\infty} Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit.
Hint: Consider $Z_n = X_n / \prod_{i=1}^{n-1}(1 + Y_i), n \geq 1$.