Stochastics II **Universität Ulm**
WS 2016/2017
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Exercise sheet 14 (total – 16 points) till February 14, 2017

Exercise 14-1 (4 points)

Let $(\Omega, \mathcal{F}, P) = ([0,1), \mathcal{B}([0,1)), \nu)$, where ν denotes the Lebesgue measure on $[0,1)$. Let $\lambda \in (0,1)$.

1. Show that $T(x) = (x+\lambda) \pmod 1$ is a measure preserving map, where $a \pmod b := a - \left\lfloor \frac{a}{b} \right\rfloor \cdot b$ for $a \in \mathbb{R}$ and $b \in \mathbb{Z}$.
2. Show that $T(x) = \lambda x$ and $T(x) = x^2$ are not measure preserving.

Exercise 14-2 (2 points)

Let a stationary sequence $X_n, n \geq 0$ be generated by a random variable X_0 and a measure preserving map T. Assume that X is m-dependent, that is, families of random variables $\{X_k, k \leq n\}$ and $\{X_j, j \geq n+m\}$ are independent for any n. Prove that T is ergodic.

Exercise 14-3 (2 points)

Let X be a non-negative random variable on some probability space $(\Omega, \mathcal{F}, P)$ and $T : \Omega \to \Omega$ a measure preserving map. Show[1] that $E(X) = E(X \circ T)$, i.e.

$$\int_\Omega X(T(\omega))P(d\omega) = \int_\Omega X(\omega)P(d\omega).$$

Exercise 14-4 (6 points)

Let $\Omega = \mathbb{R}^2$ and P be a normal distribution in $\mathbb{R}^2$ with zero mean and identity matrix of covariances. Assume that transformation $T : \Omega \to \Omega$ acts in polar coordinates as $T((r,\varphi)) = (r, 2\varphi \ (mod\ 2\pi))$, $r \geq 0, 0 \leq \varphi < 2\pi$.

1. (2 point) Prove that T preserves the measure P.
2. (4 points) Find the limit

$$\lim_{n\to\infty} \frac{1}{n} \left(\sum_{k=0}^{n-1} f(T^k(x)) \right), \ x \in \mathbb{R}^2$$

for $f_1 = x_1^2$, $f_2(x) = x_1, x_2$.

Hint: At first, prove this fact for the functions of the form
$f(r,\varphi) = \sum_{k=0}^{m} c_k \mathbb{I}\{\varphi \in [\alpha_k, \beta_k]\}\mathbb{I}\{r \in [x_k, y_k]\}$, and then pass to a limit.

Exercise 14-5 (2 points)

Let $X_n, n \geq 0$ be a cenetered Gaussian stationary sequence with covariance function $C(n) = \mathbf{E}(X_k X_{k+n})$. Let $C(n) \to 0, n \to \infty$. Prove that the measure preserving map T, which corresponds to X, i.e. $X_n \stackrel{d}{=} X_0(T^n)$, is mixing (on average) and, consequently, ergodic.

[1] Use algebraic induction.