Stochastics II **Universität Ulm**
WS 2016/2017 Prof. Dr. Evgeny Spodarev
January 16, 2017 Dr. Vitalii Makogin

Exercise sheet 11 (total – 22 points) till January 18, 2016

Exercise 11-1 (3 points)

Let $X = \{X(t), t \geq 0\}$ be a Lévy process. Show that the random variable $X(t)$ is then infinitely divisible for every $t \geq 0$.

Exercise 11-2 (2 points)

Show that the sum of two independent Lévy processes is again a Lévy process, and state the corresponding Lévy characteristic.

Exercise 11-3 (9 points)

Let X be an infinitely divisible random variable[1] with Lévy characteristics (a, b, ν).

1. Let $c : \mathbb{R} \to \mathbb{R}$ be an arbitrary function with the following asymptotical properties:

$$c(x) = 1 + o(|x|), \quad |x| \to 0 \tag{1}$$

$$c(x) = \mathcal{O}(1/|x|), \quad |x| \to \infty \tag{2}$$

   Show that the characteristic function φ of X can be written in the form

$$\varphi(z) = \exp\left[-\frac{1}{2}\tilde{a}z^2 + iz\tilde{b}_c + \int\limits_{\mathbb{R}} \left(e^{izx} - 1 - izxc(x)\right)\nu(dx)\right]$$

   where $\tilde{a} \geq 0$ and $\tilde{b}_c \in \mathbb{R}$.

2. Show that the following functions $\rho, \sigma, \tau : \mathbb{R} \to \mathbb{R}$ fullfill the conditions (1) and (2):

$$\rho(x) = 1/(1 + x^2), \quad \sigma(x) = \sin(x)/x, \quad \tau(x) = \begin{cases} 1 & ; \ |x| \leq 1 \\ 1/|x| & ; \ |x| > 1 \end{cases}$$

3. Find three more examples (different from ρ, σ, τ) for a function $c : \mathbb{R} \to \mathbb{R}$ which fullfill the conditions (1) and (2).

Exercise 11-4 (4 points)

Let $P(s) = \sum_{k=0}^{\infty} p_k s^k$ were $p_k \geq 0$ and $\sum_{k=0}^{\infty} p_k = 1$. Assume $P(0) = p_0 > 0$ and that $\log(\frac{P(s)}{P(0)})$ is a power series with positive coefficients. If φ is the characteristic function of an arbitrary distribution F show that $P(\varphi)$ is an ID characteristric function.

Hint: Find the Levy measure in terms of F^{*n}.

Exercise 11-5 (4 points)

Let φ be a characteristic function. Show that $\psi : \mathbb{R} \to \mathbb{C}$ defined by

$$\psi(z) = \frac{1-b}{1-a}\frac{1-a\varphi(z)}{1-b\varphi(z)}, 0 \leq a < b < 1,$$

is an ID characteristic function.

[1] A random variable is infinitely divisible if and only if its characteristic function admits a Lévy-Khintchine representation.