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Exercise sheet 3 (total – 17 points) till November 9, 2016

Let $\{X(t), t \in \mathbb{R}_+\}$ be a real-valued stochastic process. Denote by $\mu_X(t) = \mathbf{E}X(t), t \in \mathbb{R}_+$, $K_X(t,s) = \mathbf{Cov}(t,s), t, s \in \mathbb{R}_+$, the mean and covariance functions of the process X, respectively.

Exercise 3-1 (3 points)

Find μ_x and K_X of the process $X(t) = \xi_1 f_1(t) + \cdots + \xi_n f_n(t), t \in \mathbb{R}$, where $f_1, \ldots, f_n$ are nonrandom functions, and $\xi_1, \ldots, \xi_n$ are non-correlated random variables with means $\mu_1, \ldots, \mu_n$ and variances $\sigma_1, \ldots, \sigma_n$.

Exercise 3-2 (3 points)

Let $\{X(t), t \in \mathbb{R}_+\}$ be a stochastic process with independent increments and $\mathbf{E}|X(t)|^2 < \infty, t \in \mathbb{R}_+$. Prove that its covariance function is equal to $K_X(t,s) = F(t \wedge s), t, s \in \mathbb{R}_+$, where F is some non-decreasing function.

Exercise 3-3 (3 points)

Let X, Y be two independent square integrable centered stochastic processes and $c > 0$ be a constant. Prove that $K_{X+Y} = K_X + K_Y$, $K_{\sqrt{c}X} = cK_X$, $K_{XY} = K_X K_Y$.

Exercise 3-4 (4 points)

Let $\{X(t), t \in \mathbb{R}_+\}$ be a real-valued process with stationary and independent increments, and $\mathbf{Var}(X(1) - X(0)) > 0$. Prove that the processes X and $Y(t) := X(t+1) - X(t)$ are mean square continuous but not mean square differentiable.

Hint: You may use the solution of Cauchy's functional equation $f(x+y) = f(x) + f(y)$. If f is bounded on any interval then solution has a form $f(x) = cx$, where $c \in \mathbb{R}$.

Exercise 3-5 (4 points)

Let the mean and covariance functions of the process X be equal to $\mu_X(t) = t^2, K_X(t,s) = e^{ts}$. Prove that X is mean square differentiable. Let X' be its derivative defined in L_2 sense. Please find:

1. $\mathbf{E}[X(t) + X(s)]^2, t, s \in \mathbb{R}$.
2. $\mathbf{E}[X(t)X'(s)], t, s \in \mathbb{R}$.