Universität Ulm
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Exercise sheet 6 (total – 17 points) till November 30, 2016

Exercise 6-1 (2 points)

Prove that the distribution of a Gaussian random function X is uniquely determined by its mean value function $\mu(t) = \mathbf{E}X(t), t \in T$, and covariance function $C(s,t) = \mathbf{E}[X(s)X(t)], s,t \in T$, respectively.

Exercise 6-2 (4 points)

The fractional Brownian motion with Hurst index $H \in (0,1)$ is the centered Gaussian process $\{B^H(t), t \in \mathbb{R}\}$ with covariance $C_{B^H}(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), t,s \in \mathbb{R}$. Prove that the fractional Brownian motion (in particular, the Wiener process) is a process with stationary increments, that is, the process $\{X(s) := B^H(t+s) - B^H(t), s \geq 0\}$ is stationary.

Exercise 6-3 (2 points)

The Ornstein–Uhlenbeck process is the centered Gaussian process $\{X(t), t \in \mathbb{R}\}$ with covariance $C_X(t,s) = e^{-|t-s|}, s,t \in \mathbb{R}$. Prove that $Y(t) = \sqrt{t}X(\frac{1}{2}\log(t)), t > 0$ is the Wiener process.

Exercise 6-4 (4 points)

Prove that the processes from exercises 6-2,6-3 have continuous modifications. Find values of γ such that these processes have modifications with their trajectories satisfying the Hölder condition with index γ.

Exercise 6-5 (2 points)

The Brownian bridge is the centered Gaussian process $\{B(t), t \in [0,1]\}$ with covariance $C_B(t,s) = t \wedge s - st, s,t \in [0,1]$. Let $\{W(t), t \in [0,1]\}$ be the Wiener process. Prove that the process $\{X(t) = W(t) - tW(1), t \in [0,1]\}$ is the Brownian bridge.

Exercise 6-6 (3 points)

Let $\{B(t), t \in [0,1]\}$ be the Brownian bridge and random variable $\xi \in N(0,1)$ be independent of B. Prove that $W(t) = B(t) + t\xi$ is the Wiener process.