Universität Ulm
Prof. Dr. Evgeny Spodarev
Dr. Vitalii Makogin

Exercise sheet 7 (total – 22 points) till December 7, 2016

Exercise 7-1 (3 points)

Assume that $\zeta, \eta_1, \eta_2, \ldots$ are independent random variables, ζ takes values in $\mathbb{Z}_+$, $\mathbf{P}(\zeta = k) = p_k$, and random variables $\{\eta_k, k \in \mathbb{N}\}$ are i.i.d. and nonnegative. Let $\hat{l}_\eta(s)$ be the Laplace transform of η_1. Find the Laplace transform of $\sum_{k=1}^{\zeta} \eta_k$.

Exercise 7-2 (4 points)

Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a compound Poisson process with U_i i.i.d., $U_1 \sim Exp(\gamma)$, where the intensity of $N(t)$ is given by λ. Show that for the Laplace transform $\hat{l}_{X(t)}(s)$ of $X(t)$ it holds:

$$\hat{l}_{X(t)}(s) = \exp\left\{-\frac{\lambda t s}{\gamma + s}\right\}.$$

Exercise 7-3 (6 points)

A router has accepted n independent packets, and processes them in consecutive order. Assume that each packet has a type A with probability p and type B with probability $q = 1 - p$, and processing times have distribution functions F_A and F_B, respectively.
(a) Find the Laplace transform of a time duration necessary for all packets' service.
Particularly, find answers in the cases when
(b) F_A and F_B have exponential distribution with parameters α and β, respectively,
(c) F_A and F_B are distribution functions of constant random variables equal to α and β, respectively.

Exercise 7-4 (6 points)

Service time of a spare part has an exponential distribution with a parameter α. A processing of a new spare part starts immediately after the previous spare part was completely processed. However, if some detail is processing for more than a period of time β, then the machine overheats and stops. Let ξ be a moment when the machine stops. Prove that its distribution function F_ξ satisfies the following equation:

$$F_\xi(x) = e^{-\alpha\beta}\mathbb{I}\{x \geq \beta\} + \int_0^x F_\xi(x-y)\alpha e^{-\alpha y}\mathbb{I}\{y \in [0, \beta)\}dy.$$

Find $\mathbf{E}\xi$, $\mathbf{Var}\xi$. The service of the first spare part starts at the moment $t = 0$.

Hint: Write down an equation for $\bar{F}_\xi(x) = 1 - F_\xi(x)$. And then integrate the left and right hand sides to get $\mathbf{E}\xi$.

Exercise 7-5 (3 points)

Let $N = \{N(B), B \in \mathcal{B}(\mathbb{R}^d), \nu_d(B) < \infty\}$ be a non-homogeneous Poisson random measure with intensity measure μ.

Let $n \in \mathbb{N}, k_1 \ldots, k_n \in \mathbb{N}_0$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$ pairwise disjoint, $\nu_d(B_i) < \infty$.

1. Write down the finite dimensional distributions of N for disjoint bounded Borel sets $B_1, \ldots, B_n$.
2. Verify that for $k = \sum_{i=1}^n k_i$ and $B = \cup_{i=1}^n B_i$ it holds

$$\mathbf{P}(N(B_1) = k_1, \ldots, N(B_n) = k_n | N(B) = k) = \frac{k!}{k_1! \cdots k_n!} \frac{\mu^{k_1}(B_1) \cdots \mu^{k_n}(B_n)}{\mu^k(B)}$$

provided that $\mu(B) > 0$.